CH: 3#

Angular Momentum, K.E, Moment Inertia, Dyadics Momental Ellipsoid, Principal Axes, Euler's Angles and Euler's Equations, Conservation Theorems

Dyad and dyadic

Dyad is a simply a pair of vectors, written in a definite order without putting cross or dot between the vectors. e.g

$$\underline{A}\,\underline{B}$$

Antecedent ($\underline{A}$) → Consequent ($\underline{B}$)

Any sum of dyads is called dyadic

Nonion Form of Dyad

Let $\underline{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\underline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$$\underline{A}\underline{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= A_xB_x\,\hat{i}\hat{i} + A_xB_y\,\hat{i}\hat{j} + A_xB_z\,\hat{i}\hat{k}$$
$$+ A_yB_x\,\hat{j}\hat{i} + A_yB_y\,\hat{j}\hat{j} + A_yB_z\,\hat{j}\hat{k}$$
$$+ A_zB_x\,\hat{k}\hat{i} + A_zB_y\,\hat{k}\hat{j} + A_zB_z\,\hat{k}\hat{k}$$

This is called nonion form of the dyad, so named from the nine co-efficients involved.

This form is often written as in matrix form

$$\underline{A}\,\underline{B} = \begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix}$$

Dyadic

The sum of dyads is called dyadic. Thus the dyadic

$$\underline{P} = \underline{a}_1 \underline{b}_1 + \underline{a}_2 \underline{b}_2 + \cdots + \underline{a}_n \underline{b}_n$$

$$= \sum_{i=1}^{n} \underline{a}_i \underline{b}_i$$

represents a general dyadic in which the vectors $\underline{a}_i$ are antecedents & $\underline{b}_i$ are consequents. The dyadic

$$\underline{P}_c = \underline{b}_1 \underline{a}_1 + \underline{b}_2 \underline{a}_2 + \cdots + \underline{b}_n \underline{a}_n$$

$$= \sum_{i=1}^{n} \underline{b}_i \underline{a}_i$$

called Conjugate of $\underline{P}$

Equal Dyadics

Two dyadics $\underline{P}$ and $\underline{Q}$ are said to be equal when both transform an arbitrary vector in exactly the same way

$\underline{P} = \underline{Q}$ when and only when

$$\underline{u} \cdot \underline{P} = \underline{u} \cdot \underline{Q}$$

Symmetric and Skew dyadics#

A dyadic $\underline{P}$ is symmetric if

$$\underline{P} = \underline{P}_c$$

i.e if $\underline{P}$ & $\underline{P}_c$ transform any vector in the same manner.

$\underline{P}$ is said to be skew if

$$\underline{P} = -\underline{P}_c$$

Symmetric and skew dyadics are especially important since any dyadic $\underline{P}$ can be expressed as a sum of a symmetric and a skew dyadic in exactly one way namely

$$\underline{P} = \frac{\underline{P}+\underline{P}_c}{2} + \frac{\underline{P}-\underline{P}_c}{2}$$

Also $\underline{P} = \underline{P}_{cc}$

<u>Note</u> When vectors are cross-multiplied a dyadic $\underline{P} = \sum_{i=1}^{n} \underline{a}_i \underline{b}_i$, new dyadic formed. Thus we have

$$\underline{\lambda} \times \underline{P} = \sum \underline{\lambda} \times \underline{a}_i \underline{b}_i = \sum ($$

$$\underline{P} \times \underline{\lambda} = \sum \underline{a}_i (\underline{b}_i \times \underline{\lambda})$$

Dyad# $\hat{i}\hat{j}$

If we adjoin two vectors $\hat{i}$ & $\hat{j}$ to form the combination $\hat{i}\hat{j}$, we have a dyad. Multiplication (scalar or vector) from the left involves left hand member of pair and leaves the right-hand member strictly alone

$$\underline{A}\cdot\hat{i}\hat{j} = [((\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)\cdot\hat{i}]\hat{j}$$

$$= A_x\hat{j} \longrightarrow (1)$$

$$\hat{i}\hat{j}\cdot\underline{A} = \hat{i}A_y \longrightarrow (2)$$

Again

$$\underline{A}\times\hat{i}\hat{j} = (\underline{A}\times\hat{i})\hat{j}$$

$$= [(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)\times\hat{i}]\hat{j}$$

$$= (-\hat{k}A_y + A_z\hat{j})\hat{j}$$

$$= (A_z\hat{j} - \hat{k}A_y)\hat{j} \longrightarrow (3)$$

$$\hat{i}\hat{j}\times\underline{A} = \hat{i}(\hat{j}\times\underline{A})$$

$$= \hat{i}(-\hat{k}A_x + \hat{i}A_z) \longrightarrow (4)$$

We see that in general, the operation of multiplication is non-commutative. Note that $\hat{i}, \hat{j}$ of the dyad are not operating on each other. If they had scalar co-efficients these would be multiplied together but as far as $\hat{i}, \hat{j}$ are concerned, they are just sitting in their order. $\hat{i}\hat{j} \neq \hat{j}\hat{i}$

Unit dyad or Idemfactor

The dyad $\underline{I} = \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}$ is called unit dyad or idemfactor (idem means same latin) because it transforms any vector $\underline{A}$ into itself.

$\underline{A}\cdot\underline{I} = \underline{I}\cdot\underline{A}$

which can be proved as

Let $\underline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\underline{A}\cdot\underline{I} = \underline{A}\cdot(\hat{i}\hat{i}+\hat{j}\hat{j}+\hat{k}\hat{k})$$

$$= \underline{A}\cdot\hat{i}\hat{i} + \underline{A}\cdot\hat{j}\hat{j} + \underline{A}\cdot\hat{k}\hat{k}$$

$$= (\underline{A}\cdot\hat{i})\hat{i} + (\underline{A}\cdot\hat{j})\hat{j} + (\underline{A}\cdot\hat{k})\hat{k}$$

$$= A_1\hat{i} + A_2\hat{j} + A_3\hat{k} \qquad \because \underline{A}\cdot\hat{i} = A_1,\ \underline{A}\cdot\hat{j} = A_2,\ \underline{A}\cdot\hat{k} = A_3$$

$$= \underline{A} \longrightarrow (1)$$

$$\underline{I}\cdot\underline{A} = \underline{I}\cdot\underline{A}$$

$$= (\hat{i}\hat{i}+\hat{j}\hat{j}+\hat{k}\hat{k})\cdot\underline{A}$$

$$= \hat{i}\hat{i}\cdot\underline{A} + \hat{j}\hat{j}\cdot\underline{A} + \hat{k}\hat{k}\cdot\underline{A}$$

$$= \hat{i}(\hat{i}\cdot\underline{A}) + \hat{j}(\hat{j}\cdot\underline{A}) + \hat{k}(\hat{k}\cdot\underline{A})$$

$$= A_1\hat{i} + A_2\hat{j} + A_3\hat{k} = \underline{A} \longrightarrow (2)$$

By (1) & (2)

$\underline{A}\cdot\underline{I} = \underline{I}\cdot\underline{A}$ Proved.

The Scalar Dot product of a dyad with vector

The scalar dot product of dyad $\underline{A}\,\underline{B}$ with vector is defined as

$$\underline{A}\,\underline{B}\cdot\underline{C} = \underline{A}(\underline{B}\cdot\underline{C}) \longrightarrow (1)$$

$$\underline{C}\cdot\underline{A}\,\underline{B} = (\underline{C}\cdot\underline{A})\underline{B} \longrightarrow (2)$$

In (1) $\underline{C}$ is called postfactor and in (2) $\underline{C}$ is called the prefactor.

Problem # Prove that dyad scalar multiplication with a vector is not commutative in general

Sol # Let $\underline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$\underline{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$\underline{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$

In nonion form

$$\underline{A}\,\underline{B} = A_1B_1\hat{i}\hat{i} + A_1B_2\hat{i}\hat{j} + A_1B_3\hat{i}\hat{k} + A_2B_1\hat{j}\hat{i} + A_2B_2\hat{j}\hat{j} + A_2B_3\hat{j}\hat{k} + A_3B_1\hat{k}\hat{i} + A_3B_2\hat{k}\hat{j} + A_3B_3\hat{k}\hat{k}$$

$$\underline{A}\,\underline{B}\cdot\underline{C} = A_1B_1\hat{i}(\hat{i}\cdot\underline{C}) + A_1B_2\hat{i}(\hat{j}\cdot\underline{C}) + A_1B_3\hat{i}(\hat{k}\cdot\underline{C}) + A_2B_1\hat{j}(\hat{i}\cdot\underline{C}) + A_2B_2\hat{j}(\hat{j}\cdot\underline{C}) + A_2B_3\hat{j}(\hat{k}\cdot\underline{C}) + A_3B_1\hat{k}(\hat{i}\cdot\underline{C}) + A_3B_2\hat{k}(\hat{j}\cdot\underline{C}) + A_3B_3\hat{k}(\hat{k}\cdot\underline{C})$$

$$= A_1B_1C_1\hat{i} + A_1B_2C_2\hat{i} + A_1B_3C_3\hat{i} + A_2B_1C_1\hat{j} + A_2B_2C_2\hat{j} + A_2B_3C_3\hat{j} + A_3B_1C_1\hat{k} + A_3B_2C_2\hat{k} + A_3B_3C_3\hat{k} \rightarrow (1)$$

$$\underline{C}\cdot\underline{A}\,\underline{B} = (C_1A_1B_1 + C_2A_2B_1 + C_3A_3B_1)\hat{i} + (C_1A_1B_2 + C_2A_2B_2 + C_3A_3B_2)\hat{j} + (C_1A_1B_3 + C_2A_2B_3 + C_3A_3B_3)\hat{k} \rightarrow (2)$$

from (1) & (2)

$$\underline{A}\,\underline{B}\cdot\underline{C} \neq \underline{C}\cdot\underline{A}\,\underline{B}$$

Note: We note from (1) & (2) that a scalar dot product of dyad with a vector is a vector ... that dyad scalar multiplication ... not commutative ...

Double Dot Product of Two Dyads

$$\underline{A}\,\underline{B} : \underline{C}\,\underline{D} = (\underline{A}\cdot\underline{C})(\underline{B}\cdot\underline{D})$$

A more convenient notation is to write the double dot product as

$$\underline{A}\,\underline{B} : \underline{C}\,\underline{D} \equiv (\underline{C}\cdot\underline{A})(\underline{B}\cdot\underline{D})$$

$$= \underline{C}\cdot\underline{A}\,\underline{B}\cdot\underline{D}$$

Here we note that $\underline{C}$ becomes prefactor and $\underline{D}$ becomes postfactor

Dot and Cross-Products in dyadic Form

In view of double dot product of two dyads as defined above, we can define dot and cross-products of vectors as under

We know that

$$\underline{A}\cdot\underline{B} = A_1B_1 + A_2B_2 + A_3B_3 \rightarrow (1)$$

Now taking dyad $\underline{A}\,\underline{B}$ and double dot product with dyad $\hat{i}\hat{i}$ as

$$\underline{A}\,\underline{B} : \hat{i}\hat{i} = \hat{i}\cdot\underline{A}\,\underline{B}\cdot\hat{i}$$

$$= (\hat{i}\cdot\underline{A})(\underline{B}\cdot\hat{i}) = A_1B_1$$

Similarly

$$\underline{A}\,\underline{B} : \hat{j}\hat{j} = (\hat{j}\cdot\underline{A})(\underline{B}\cdot\hat{j}) = A_2B_2$$

$$\underline{A}\,\underline{B} : \hat{k}\hat{k} = (\hat{k}\cdot\underline{A})(\underline{B}\cdot\hat{k}) = A_3B_3$$

using in (1)

$$\underline{A}\cdot\underline{B} = \hat{i}\cdot(\underline{A}\underline{B})\cdot\hat{i} + \hat{j}\cdot(\underline{A}\underline{B})\cdot\hat{j} + \hat{k}\cdot(\underline{A}\underline{B})\cdot\hat{k}$$

$$\underline{A} \times \underline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \hat{i}\,[A_2 B_3 - A_3 B_2] + \hat{j}\,[A_3 B_1 - A_1 B_3]$$
$$+ \hat{k}\,[A_1 B_2 - A_2 B_1] \rightarrow ①$$

Now

$$\underline{A}\underline{B} : \hat{j}\hat{k} = \hat{j} \cdot \underline{A}\,\underline{B} \cdot \hat{k}$$
$$= (\hat{j} \cdot \underline{A})\,(\underline{B} \cdot \hat{k}) = A_2 B_3$$
$$\Rightarrow \hat{j} \cdot (\underline{A}\underline{B}) \cdot \hat{k} = A_2 B_3$$

Similarly,

$$A_3 B_2 = \hat{k} \cdot (\underline{A}\,\underline{B}) \cdot \hat{j}$$
$$A_3 B_1 = \hat{k} \cdot (\underline{A}\,\underline{B}) \cdot \hat{i}$$
$$A_1 B_3 = \hat{i} \cdot (\underline{A}\,\underline{B}) \cdot \hat{k}$$
$$A_1 B_2 = \hat{i} \cdot (\underline{A}\,\underline{B}) \cdot \hat{j}$$
$$A_2 B_1 = \hat{j} \cdot (\underline{A}\,\underline{B}) \cdot \hat{i}$$

using these in ①

$$\underline{A} \times \underline{B} = \hat{i}\,[\hat{j} \cdot (\underline{A}\underline{B}) \cdot \hat{k} - \hat{k} \cdot (\underline{A}\,\underline{B}) \cdot \hat{j}]$$
$$+ \hat{j}\,[\hat{k} \cdot (\underline{A}\underline{B}) \cdot \hat{i} - \hat{i} \cdot (\underline{A}\underline{B}) \cdot \hat{k}]$$
$$+ \hat{k}\,[\hat{i} \cdot (\underline{A}\underline{B}) \cdot \hat{j} - \hat{j} \cdot (\underline{A}\,\underline{B}) \cdot \hat{i}]$$

which is cross product in dyadic form.

Behaviour of Dyad as Tensor

In nonion form dyad $\underline{A}\underline{B}$ can be written as

$$\underline{A}\underline{B} = \begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{pmatrix}$$

If A_i & B_j are components of vectors $\underline{A}$ & $\underline{B}$, then $C_{ij} = A_iB_j$, being outer product of vectors which is obtained by adjoining directly the components of vectors $\underline{A}$ & $\underline{B}$. So co-efficients of the nonion representation of a dyadic transform under an orthogonal transformation exactly as do the components of a 2nd rank tensor. There is also an equivalence in their effect as operators acting on vectors because we have seen that the dot product of a dyad or a dyadic with a vector result in a new vector just dot product of a 2nd rank tensor with a vector gives a tensor of rank one (vector). A dyadic is therefore in all ways equivalent to a tensor of 2nd rank

The Dyadic Product $\underline{P}\cdot\underline{Q}$

If $\underline{P}$ & $\underline{Q}$ are dyadics, then product $\underline{P}\cdot\underline{Q}$ is defined by

$$(\underline{P}\cdot\underline{Q})\cdot\underline{U} = \underline{P}\cdot(\underline{Q}\cdot\underline{U}) \quad \text{for every vector } \underline{U}$$

From this definition ...

Associative $(\underline{P}\cdot\underline{Q})\cdot\underline{R} = \underline{P}\cdot(\underline{Q}\cdot\underline{R})$

Distributive $(\underline{P}+\underline{Q})\cdot\underline{R} = \underline{P}\cdot\underline{R} + \underline{Q}\cdot\underline{R}$

But in general $\underline{P}\cdot\underline{Q} \neq \underline{Q}\cdot\underline{P}$

<u>Problem</u># The dot multiplication of a dyadic with vector is Commutative iff the dyadic is symmetric

<u>Sol</u> Let $\underline{P} = \sum_{i=1}^{n} \underline{a}_i\underline{b}_i$ be symmetric

dyadic, then

$$\underline{P} = \underline{P}_c$$

$$\Rightarrow \sum a_i b_i = \sum \underline{b}_i\underline{a}_i$$

For any vector $\underline{u}$

$$\underline{P}\cdot\underline{U} = \sum \underline{a}_i\underline{b}_i\cdot\underline{u}$$

$$= \sum \underline{b}_i\underline{a}_i\cdot\underline{u}$$

$$= \underline{P}_c\cdot\underline{U}$$

$$= \underline{P}\cdot\underline{u}$$

$$\underline{P}\cdot\underline{U} = \sum \underline{a}_i(\underline{b}_i\cdot\underline{u})$$

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Note# (1)# $(\underline{a}\,\underline{b})\cdot(\underline{c}\,d) = (\underline{b}\cdot\underline{c})\,a\underline{d}$

(2)# $(\underline{P}\cdot\underline{Q})_c = \underline{Q}_c\cdot\underline{P}_c$

<u>Problem</u># If $\underline{P}^n = \underline{P}\cdot\underline{P}\cdot\underline{P}\cdot\ldots$ to n factors

Prove that (i) $\underline{I}\times\hat{\underline{k}} = \hat{j}\hat{i} - \hat{i}\hat{j}$ (ii) $(\underline{I}\times\hat{k})^2 = -\hat{i}\hat{i} - \hat{j}\hat{j}$

(iii) [illegible] $(\underline{I}\times\hat{k})^4 = \hat{i}\hat{i} + \hat{j}\hat{j}$

A Property of Symmetric Dyadic

One of the most significant properties of a symmetric dyadic is that it can always be put in normal or diagonal form by proper choice of Co-ordinate axes

$$\underline{T} \rightarrow \hat{i}\hat{i}T_{xx} + \hat{j}\hat{j}T_{yy} + \hat{k}\hat{k}T_{zz}$$

all the non-diagonal elements going to zero. The Co-ordinate transformation that put the dyadic in this diagonal form is known as the principal axis transformation.

There is a useful geometric interpretation of a symmetric dyadic

For simplicity let us suppose our symmetric dyadic is already in its diagonal form. Then, with $\underline{r}$, the usual distance vector, we form the equation

$$\underline{r} \cdot \underline{T} \cdot \underline{r} = 1 \longrightarrow (1)$$

which limits the length of $\underline{r}$ according to its orientation.

By expanding (1)

$$(\hat{i}x + \hat{j}y + \hat{k}z) \cdot (\hat{i}\hat{i}T_{xx} + \hat{j}\hat{j}T_{yy} + \hat{k}\hat{k}T_{zz}) \cdot (\hat{i}x + \hat{j}y + \hat{k}z) = 1$$

$$x^2 T_{xx} + y^2 T_{yy} + z^2 \text{[illegible]}$$

$$\frac{x^2}{\frac{1}{T_{xx}}} + \frac{y^2}{\frac{1}{T_{yy}}} + \frac{z^2}{\frac{1}{T_{zz}}} = 1$$

$$\frac{x^2}{\left(\sqrt{\frac{1}{T_{xx}}}\right)^2} + \frac{y^2}{\left(\sqrt{\frac{1}{T_{yy}}}\right)^2} + \frac{z^2}{\left(\sqrt{\frac{1}{T_{zz}}}\right)^2} = 1$$

This is an ellipsoid with semi-axes

$a = T_{xx}^{-1/2}$ $b = T_{yy}^{-1/2}$ $c = T_{zz}^{-1/2}$

Thus diagonalizing our dyadic, gives such an ellipsoid (dyadic ellipsoid), whose axes are are lined up with the co-ordinate axes.

Antisymmetric Dyadic

If $\underline{U}$ is an antisymmetric dyadic, then

$$U_{xx} = 0 \quad \text{etc}$$

$$U_{xy} = -U_{yx} \quad \text{etc}$$

Then for any vector $\underline{a}$

$$\underline{a} \cdot \underline{U} = -\underline{U} \cdot \underline{a}$$

$\Rightarrow$ Multiplication of a vector and anti-symmetric dyadic obey an anti-commutation rule.

Note Dyadis are rather awkward to handle in comparison with tensor anaylsis. They are difficult to control for representing third or higher rank tensors.

The Invariants of a Dyadic#

The dyadic $\underline{P} = \sum \underline{a}_i \underline{b}_i$ has scalar invariant $P_s = \sum \underline{a}_i \cdot b_i$

and vector invariant $\underline{P} = \sum_{i=1}^{n} \underline{a}_i \times b_i$

If Two dyadics are equal, then their scalar and vector invariants are equal

The scalar and vector invariants of the sum of two dyadics are the sum of their respective invariants.

If $\underline{R} = \underline{P} + \underline{Q}$

Then

$R_s = P_s + Q_s$ $\quad \underline{R} = \underline{P} + \underline{V}$

It is this property that gives these invariants their special importance in geometry and physics.

Problem# If $\underline{P}^n = \underline{P}\cdot\underline{P}\cdot\underline{P} \cdots \cdots \underline{P}$ to n factors

(i) $\underline{I} \times \hat{k} = \hat{j}\hat{i} - \hat{i}\hat{j}$ $\quad$ (ii) $(\underline{I} \times \hat{k})^2 = -\hat{i}\hat{i} - \hat{j}\hat{j}$

(iii) $(\underline{I} \times \hat{k})^3 = \hat{i}\hat{j} - \hat{j}\hat{i}$ $\quad$ (iv) $(\underline{I} \times \hat{k})^4 = \hat{i}\hat{i} + \hat{j}\hat{j}$

Sol# (i) $(\underline{I} \times \hat{k}) = (\hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}) \times \hat{k}$

$= \hat{i}\,\hat{i} \times \hat{k} + \hat{j}\,\hat{j} \times \hat{k} + \hat{k}\,\hat{k} \times \hat{k}$

$= -\hat{i}\hat{j} + \hat{j}\hat{i} + 0 = \hat{j}\hat{i} - \hat{i}\hat{j}$

(ii) $(\underline{I} \times \hat{k})^2 = (\underline{I} \times \hat{k}) \cdot (\underline{I} \times \hat{k})$

$= (\hat{j}\hat{i}$ [illegible]

$= \hat{j}\hat{i}\cdot\hat{j}\hat{i} - \hat{j}\hat{i}\cdot\hat{i}\hat{j} - \hat{i}\hat{j}\cdot\hat{j}\hat{i} + \hat{i}\hat{j}\cdot\hat{i}\hat{j}$

$= 0 - \hat{j}\hat{j} - \hat{i}\hat{i} + 0$

$= -\hat{j}\hat{j} - \hat{i}\hat{i}$

Similarly other parts can be proved

Matrices and Tensors

The product AB of two matrices exists only if the number of Columns of A is equal to number rows of B.

In view of this the product of a square matrix with a single column matrix (or vector column matrix) can be formed. A single row matrix can indeed pre multiply a square matrix. A symbol $\underline{x}$ can be used to express a single column matrix or a row matrix and in expression $A\underline{x}$, $\underline{x}$ stands for column vector while in $\underline{x}A$, $\underline{x}$ stands for row vector.

ith Component of $A\underline{x}$ can be written as

$$A_{ij}\, x_j = x_j\, (\tilde{A})_{ji}$$

where $\tilde{A}$ is transpose of A

Hence for a square matrix A, we have a useful Commutation property of the product of a vector and a square matrix that.

$i U =$

A square matrix A is symmetric if

$$A = \tilde{A}$$

or

$$A_{ij} = A_{ji}$$

and is anti-symmetric or skew symmetric if

$$A = -\tilde{A}$$

or

$$A_{ij} = -A_{ji}$$

Clearly in an anti-symmetric matrix, the diagonal elements are always zero.

The two interpretations of an operator as transforming the vector or alternatively the co-ordinate system are both involved if we find the transformation of an operator under a change of co-ordinates.

Let A be considered an operator acting upon a vector $\underline{F}$ (or a single column matrix $\underline{F}$) to produce a new vector G

$$G = A\underline{F} \quad \longrightarrow ①$$

If the co-ordinates system is transformed by a matrix B, the components of vector G in the new system will be given by

$$BG = BA\underline{F} \quad \text{by } ① \quad ②$$

which can be written as

$$BG = BAB^{-1}B\underline{F} \quad \longrightarrow ②$$

$B\underline{F}$ gives vector $\underline{F}$ expressed in new co-ordinate system and operator BAB^{-1} gives the vector BG which is vector G expressed in the new co-ordinate system. We may consider BAB^{-1} to be the form taken by operator A when [illegible] to [illegible]

set of axes ,

$$A' = BAB^{-1} \rightarrow (3)$$

Any tranformation of matrix having the form (3) is known as similarity transformation. i.e A' is similar to A

Now the nine components of a 2nd rank tensor transform as

$$T'_{ij} = a_{ik}\, a_{jl}\, T_{kl}$$

We must distinguish between a 2nd rank tensor T and the square matrix formed from its components. A tensor is defined only in terms of its transformation properties under orthogonal co-ordinate transformations. on the other hand, a matrix is no way restricted in the types of transformations and indeed may be considered entirely independtly of its properties under some particular class of transformations. Within the domain of orthogonal transformations, there is a practical identity. The tensor components and matrix elements of are manipulated in the same fashion: for every tensor equation there will be a corresponding matrix equation and vice versa. By equation (3), the components of a square matrix T transform under a linear change co-ordinates defined by matrix A according to a similar transformation

$$T' = ATA^{-1}$$

For an orthogonal transformation, we have

$$T' = AT\tilde{A} \qquad \because \tilde{A} = A^{-1}$$

$T_{kl}\, a_{jl}$

This shows that the matrix components trans form identically, under an orthogonal transformation, with the components of a tensor of a 2nd rank.

All the terminolog and operations of matrix algebra, such as "transpose" and anti-symmetric can be applied tensor without change. The equivalence between the tensor and matrix is not restricted to tensors of 2nd rank. e.g, the components of a vector which is a tensor of ist rank, form a column or row matrix and vector manipulation may be treated completely in terms of these associated matrices.

Two vectors can be used to form a 2nd rank tensor, T; Let A & B be vectors with components A_i and B_i and form a tensor T by

$$T_{ij} = A_i B_j$$

e.g if $\underline{A}$ & $\underline{B}$ are two dimensional vectors, then

$$T = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix} = \begin{pmatrix} A_x B_x & A_x B_y \\ A_y B_x & A_y B_y \end{pmatrix}$$

Since each individual vector transform as a vector under a cartesian tranformation, each component of T will transform as a tensor

$$T'_{xy} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{xi} a_{yj} T_{ij}$$

$$= a_{xi} a_{yj} A_i B_j = a_{xi} A_i \, a_{yj} B_j = A'_x B'_y$$

so T is a tensor.

The types of operations performed with vectors can be formed with tensors. There is a unit tensor

$$I_{ij} = \delta_{ij} = 1 \quad i = j$$
$$\qquad\qquad = 0 \quad i \neq j$$

The dot product of on the R.H.S of tensor $\underline{T}$ with a vector $\underline{C}$ is defined as the vector $\underline{D}$ by

$$\underline{D} = \underline{T}\cdot\underline{C} \qquad \underline{T} = \underline{A}\,\underline{B} \text{ dyad}$$

where $D_i = \sum_{j=1}^{3} T_{ij}C_j = T_{ij}C_j$

and dot product on the left with a vector $\underline{F}$ is

$$\underline{E} = \underline{F}\cdot\underline{T}$$

where $E_i = \sum_{j=1}^{3} F_j T_{ji} = F_j T_{ji}$

A scalar s can be considered by a a double dot product

$$s = \underline{F}\cdot\underline{T}\cdot\underline{C} = \sum_{i=1}^{3}\sum_{j=1}^{3} F_i T_{ij} C_j$$

$$= F_i T_{ij} C_j$$

These processes are termed as contraction. If Tensor $\underline{T}$ is constructed of two vectors $\underline{A}$ & $\underline{B}$, then

$$\underline{T}\cdot\underline{C} = \underline{A}\,\underline{B}\cdot\underline{C} = \underline{A}(\underline{B}\cdot\underline{C})$$

and

$$\underline{F}\cdot\underline{T} = \underline{F}\cdot\underline{A}\,\underline{B} = (\underline{F}\cdot\underline{A})\,\underline{B}$$

which we have already discussed under dyadic

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Linear Momentum of a Particle

Let $\underline{r}$ be the radius vector of a particle from some given origin and $\underline{V} = \frac{d\underline{r}}{dt}$ is vector velocity.

The linear momentum $\underline{P}$ of the particle is defined as the product of particle mass and velocity

$$\underline{P} = m\underline{V} \quad \rightarrow (1)$$

Due to interaction with the external objects and fields the particle may experience forces of various types. If $\underline{F}$ is sum of these forces and it produces acceleration $\underline{a}$ in the particle, then

$$\underline{F} = m\underline{a} = m\frac{d\underline{v}}{dt} \quad \rightarrow (2)$$

Differentiating (1) w.r.t t

$$\frac{d\underline{P}}{dt} = m\frac{d\underline{V}}{dt}$$

$$\frac{d\underline{P}}{dt} = \underline{F} \quad \rightarrow (3)$$

This called equation of motion of particle or Newton's 2nd law of motion.

A frame of reference in which (3) is valid is called inertial-frame or Galilean system or Newtonian frame.

Law of Conservation of Linear Momentum of Particle

If $\underline{F}$ is resultant force on a particle of mass m and $\underline{V}$ is the linear velocity of particle, then

$$\underline{F} = \frac{d\underline{P}}{dt} \quad \rightarrow ①$$

Now if particle is free i.e resultant force on the particle is zero, Then.

$$\frac{d\underline{P}}{dt} = \underline{0}$$

Integrating

$$\underline{P} = \text{Constant}$$

$\Rightarrow$ $\underline{P}$ is a vector constant in time and the linear momentum of the free particle is conserved.

Since this result is obtained by vector equation

$$\dot{\underline{P}} = \underline{0}$$

Therefore applies for each component of linear momentum. In order to state the result in another way. Let $\underline{a}$ be some constant vector such that

$$\underline{F} \cdot \hat{\underline{a}} = 0 \quad \text{independent}$$

of time, then from ①

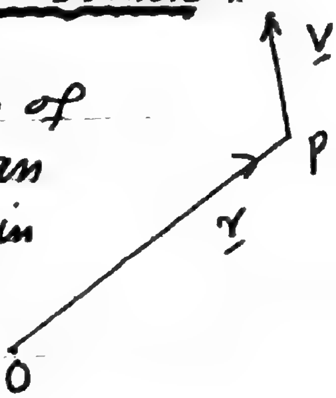
$$\frac{d\underline{P}}{dt} \cdot \hat{a} = \underline{F} \cdot \hat{a} = 0$$

$$\Rightarrow \frac{d}{dt}(\underline{P} \cdot \hat{a}) = 0$$

$$\Rightarrow \underline{P} \cdot \hat{a} = \text{Constant}$$

which states that component of linear momentum in a direction in which the force component vanishes is constant in time.

Angular Momentum of a Particle



The angular momentum of a particle referred to an arbitrary fixed point (fixed in an inertial reference point) as origin is defined as moment of linear momentum.

It is usually denoted by $\underline{L}$.

Thus

$$\underline{L} = \underline{r} \times \underline{P}$$

$$= \underline{r} \times m\underline{v}$$

Relation between Torque and Angular Momentum

&

Law of Conservation of Angular Momentum

The angular momentum $\underline{L}$ of a particle w.r.t an origin from which $\underline{r}$ is measured is defined by

$$\underline{L} = \underline{r} \times \underline{P} \longrightarrow ①$$

If the force $\underline{F}$ acts upon the particle, then torque or moment of $\underline{F}$ about the same fixed point is defined as

$$\underline{N} = \underline{r} \times \underline{F} \longrightarrow ②$$

Diff ① w.r.t t

$$\frac{d\underline{L}}{dt} = \frac{d}{dt}(\underline{r} \times \underline{p})$$

$$= \underline{r} \times \frac{d\underline{p}}{dt} + \frac{d\underline{r}}{dt} \times \underline{p}$$

$$= \underline{r} \times \frac{d\underline{p}}{dt} + \underline{v} \times m\underline{v}$$

$$= \underline{r} \times \frac{d\underline{p}}{dt} + \underline{0}$$

$$= \underline{r} \times \frac{d\underline{p}}{dt}$$

$$= \underline{r} \times \underline{F} \quad \because \text{By Newton's Law } \underline{F} = \frac{d\underline{p}}{dt}$$

$$= \underline{N} \quad \text{by ②}$$

$$\Rightarrow \dot{\underline{L}} = \underline{N}$$

The time rate of change of angular momentum of a particle is equal to the torque on the particle.

Both $\underline{N}$ and $\underline{L}$ depend upon the point o about which the moments are taken.

If there are no torques on the particles i.e if $\underline{N} = \underline{0}$, then

$$\dot{\underline{L}} = \underline{0}$$

$$\Rightarrow \underline{L} = \text{Constant}$$

$\Rightarrow$ If the total torque on the particle is zero, then angular momentum is conserved.

③

Work on Particle

If the resultant external force $\underline{F}$ translate a particle of mass m from position 1 to 2, then the work done by $\underline{F}$ during small displacement $d\underline{r}$ is

$$dW_{12} = \underline{F} \cdot d\underline{r}$$

Hence total work done from position 1 to 2 is

$$W_{12} = \int_1^2 \underline{F} \cdot d\underline{r}$$

But $\underline{F} = m\dfrac{d\underline{v}}{dt}$ $\quad d\underline{r} = \underline{v}\,dt$

$$W_{12} = \int_1^2 m\frac{d\underline{v}}{dt} \cdot \underline{v}\,dt$$

If mass of particle is constant, then

$$W_{12} = m\int_1^2 \frac{d\underline{v}}{dt} \cdot \underline{v}\,dt$$

$$= \frac{m}{2}\int_1^2 \frac{d}{dt}(\underline{v} \cdot \underline{v})\,dt$$

$$= \frac{m}{2}\int_1^2 \frac{d}{dt}(v^2)\,dt$$

and therefore

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2)$$

$$= T_2 - T_1$$

where $T = \frac{1}{2}mv^2$ is the k.E. of the particle

If $T_1 > T_2$, then W_{12} is [illegible] and particle

has done work with a resulting decrease in k.E

Angular Impulse

To obtain the effect of torque $\underline{N}$ on the angular momentum of a particle about a fixed point O (O fixed in some Newtonian frame) over a finite interval of time, we use equation

$$\frac{d\underline{L}}{dt} = \underline{N}$$

$$\underline{N}\,dt = d\underline{L}$$

$$\int_{t_1}^{t_2} \underline{N}\,dt = \underline{L}_2 - \underline{L}_1 = \Delta \underline{L} \quad \rightarrow (1)$$

where $\underline{L}_2 = \underline{r}_1 \times m\underline{v}_1$ is angular momentum at time t_1 and $\underline{L}_1 = \underline{r}_1 \times m\underline{v}_1$ is angular momentum at t_2

The product of moment (torque) and time is defined as impulse

Equation (1) states that the total angular ~~(momentum)~~ impulse on a particle m about a fixed point O is equal to the corresponding change in angular momentum.

Power

The average power Pav delivered to a particle or a body by a a resultant force $\underline{F}$ during total time t

$$= \frac{\text{total work}}{\text{total time}} = \frac{W}{t}$$

If $dw = \underline{F} \cdot d\underline{r}$ is small amount of work done in the infinitesimal time interval dt, then

$$dw = dT = \underline{F} \cdot d\underline{r}$$

Now instantaneous power P is given by

$$P = \frac{dw}{dt} = \frac{dT}{dt} = \underline{F} \cdot \frac{d\underline{r}}{dt}$$

$$= \underline{F} \cdot \underline{v}$$

If $\underline{F}$ and $\underline{v}$ are parallel, then

$$P = Fv$$

Also if power is constant, then instantaneous and average powers are equal i.e

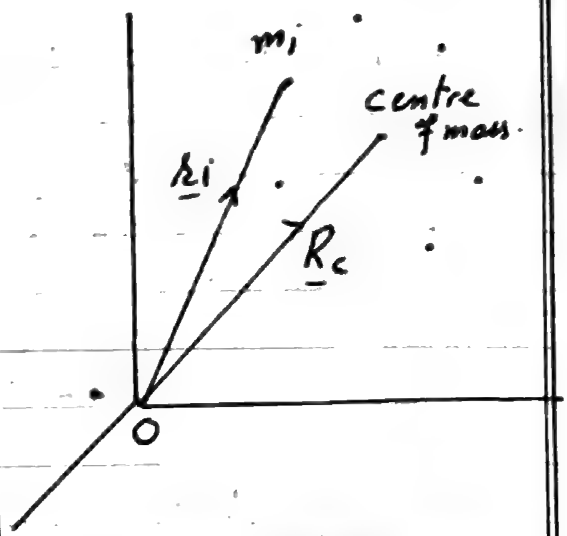
$$P_{av} = P$$

Remarks # (1) Here we have considered only mechanical power, which results from mechanical work. A more general view of power is energy delivered per unit time and this we broaden the concept of power to include electrical power, solar power and so on.

(2) If we choose a certain inertial frame of reference for the description of a mechanical process, the laws of motion are the same as in any other reference frame which is in uniform motion relative to original frame. The velocity of a particle is in general different, depending upon the frame of reference chosen as basis for the description of motion. Therefore it is impossible to ascribe an absolute K.E to a particle as such

Centre of Mass of System of Particles and its Motion

Suppose we have a system of n particles of masses $m_1, m_2, \ldots\ldots m_n$ with radii vectors $\underline{r}_1, \underline{r}_2, \ldots\ldots \underline{r}_n$ relative to given origin O. The the vector $\underline{R}_c$ defined by



$$\underline{R}_{cm} = \frac{\sum_{i=1}^{n} m_i \underline{r}_i}{\sum_{i=1}^{n} m_i}$$

$$= \frac{1}{M} \sum_{i=1}^{n} m_i \underline{r}_i$$

is called centre of mass of the system of particles and it is independent of the choice of origin i.e if we take any other fixed point as origin, the p.v of c.m will again be given by

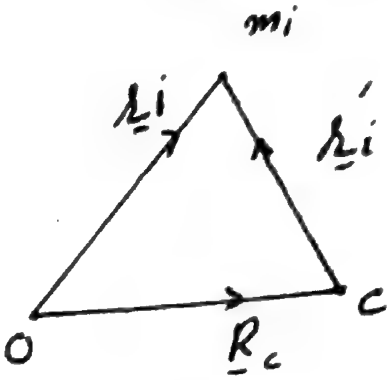
$$\underline{R}_{cm} = \frac{\sum m_i \underline{r}_i}{M}$$

Here $\sum_{i=1}^{n} m_i \underline{r}_i$ is sum of mass moment about point O

Problem # Prove that the sum of mass moments of [illegible] of particles about centre of mass is [illegible] = 0 [illegible]

Sol # Consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ at position $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_n$ relative to a fixed origin O. Let $\underline{r}'_1, \underline{r}'_2, \ldots, \underline{r}'_n$ be their p.v relative to mass centre C. Then

$$\underline{R}_{cm} = \frac{\sum m_i \underline{r}_i}{\sum m_i}$$



$$= \frac{1}{M} \sum m_i \underline{r}_i$$

$$\Rightarrow \sum m_i \underline{r}_i = M \underline{R}_{cm}$$

putting $\underline{r}_i = \underline{R}_{cm} + \underline{r}'_i$

$$\Rightarrow \sum m_i (\underline{R}_{cm} + \underline{r}'_i) = M \underline{R}_{cm}$$

$$\Rightarrow \sum m_i \underline{R}_{cm} + \sum m_i \underline{r}'_i = M \underline{R}_{cm}$$

$$\Rightarrow M \underline{R}_{cm} + \sum m_i \underline{r}'_i = M \underline{R}_{cm}$$

$$\Rightarrow \sum m_i \underline{r}'_i = \underline{0}$$

$\Rightarrow$ Sum of moments relative to mass-centre is zero

Motion of Centre of Mass

Theorem # (a) Prove that the centre of mass C, of the whole system of particles moves as if the the whole mass M of the system is concentrated at C and the resultant external force on the system is applied to M at C i.e $M\ddot{\underline{R}}_{cm} = \underline{F}$

(b) # Prove that the linear momentum of the system of particles is same as if a single particle of Mass M equal to total mass of system were located at c.m and moving with the velocity of c.m i.e

$$\underline{P} = M\dot{\underline{R}}_{cm} = M\underline{V}_{cm}$$

(c) # The rate of change of momentum of the system of particle is equal to the resultant external force on the system i.e

$$\dot{\underline{P}} = \underline{F}$$

Proof # (a) Suppose the system consists of n particles of masses with masses m_1, m_2 ---- m_n at positions $\underline{r}_1, \underline{r}_2$ --- $\underline{r}_n$ w.r.t a fixed origin O. The resultant force with acts ith particle within the system is in general composed of two parts. On part is the resultant of all forces whose origin lies outside the system; this is called external force $\underline{F}_i^{(e)}$ on ith particle. The other part is the resultant of the forces which arise from the interaction of all $(n-1)$ particles with the ith particle. This is called internal force $\underline{F}_i$, which will be the sum of the internal forces acting on ith particle i.e

$$\underline{F}_i = \sum_{j=1}^{n} \underline{F}_{ij}$$ where $\underline{F}_{ij}$ is internal force on ith particle due to jth particle $(i \neq j)$.

Thus ~~total~~ force $\underline{F}_i$ acting on the ith particle is

$$\underline{F}_i = \underline{F}_i^{(e)} + \underline{F}_i' \longrightarrow (1)$$

Newton's 2nd Law or equation of motion for the ith particle is given by

$$\dot{\underline{P}}_i = m_i \ddot{\underline{r}}_i = \underline{F}_i = \underline{F}_i^{(e)} + \underline{F}_i'$$

$$\dot{\underline{p}}_i = \underline{F}_i^{(e)} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \underline{F}_{ij} \qquad (\underline{F}_{ii} = \underline{0} \text{ naturally})$$

$$\Rightarrow \quad m_i \frac{d^2 r_i}{dt^2} = \underline{F}_i^{(e)} + \sum_{j=1}^{n} \underline{F}_{ij}$$

Summing over all the particles, we have

$$\frac{d^2}{dt^2}\left(\sum_{i=1}^{n} m_i \underline{r}_i\right) = \sum_i \underline{F}_i^{(e)} + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \underline{F}_{ij}\right) \longrightarrow (2)$$

Now we assume that $\underline{F}_{ij}$ (like $\underline{F}_i^{(e)}$) obey newton's 3rd law of motion in its original form i.e the forces which two particle exert on each other are equal and opposite. This assumption (which does not hold for all types of forces e.g for moving charged particles because electromagnetic forces are velocity dependent) is sometimes called the weak law of action and Reaction.

Now

$$\underline{F}_{ij} = -\underline{F}_{ji}$$

and

$$\sum_{i=1}^{n} \underline{F}_i' = \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{F}_{ij} = \sum_{i,j=1}^{n} \underline{F}_{ji}$$

($\because$ i, j are dummies $\therefore$ we may interchange these without effecting the sum.)

$$= \sum_{i,j=1}^{n} -\underline{F}_{ij} = -\sum_{i=1}^{n} \underline{F}_i'$$

$$\Rightarrow \sum_{i=1}^{n} \underline{F}_i' = \underline{0}$$

Also $\sum_{i=1}^{n} \underline{F}_i^{(e)}$ is the sum of all the external forces on all of the particles of the system and can be written as

$$\sum_{i=1}^{n} \underline{F}_i^{(e)} = \underline{F}^{(e)} = \underline{F}$$

Equation ② becomes

$$\frac{d^2}{dt^2}\left(\sum_{i=1}^{n} m_i \underline{r}_i\right) = \underline{F}$$

But $\underline{R}_{cm} = \dfrac{\sum m_i \underline{r}_i}{M}$

$$\Rightarrow \sum m_i \underline{r}_i = M \underline{R}_{cm}$$

Using this we have

$$\frac{d^2}{dt^2}(M\underline{R}_{cm}) = \underline{F}$$

$$M\frac{d^2 R_{cm}}{dt^2} = \underline{F}$$ if mass of system is constant

$$M\ddot{\underline{R}}_{cm} = \underline{F} \quad \rightarrow ③$$

which states that the centre of mass of the system moves as if it were a single particle, of mass equal to mass of the system, acted upon by total external force and independent of the nature of internal forces as long as they follow $\underline{F}_{ij} = -\underline{F}_{ji}$

(b) # Total linear momentum of the system is

$$\underline{P} = \sum_{i=1}^{n} m_i \dot{\underline{r}}_i = \frac{d}{dt}\left(\sum_{i=1}^{n} m_i \underline{r}_i\right)$$

$$\underline{P} = \frac{d}{dt}(M\underline{R}_{cm})$$

$$= M\frac{d\underline{R}_{cm}}{dt}$$

$$\underline{P} = M\dot{\underline{R}}_{cm} = M\underline{V}_{cm} \rightarrow (4)$$

This states the linear momentum of the system is same as if a single particle (fictitious) of mass $M = \Sigma m_i$ were placed at c.m and moving with the velocity of c.m.

(C)# Differentiating (4)

$$\frac{d\underline{P}}{dt} = M\ddot{\underline{R}}_{cm} = \underline{F} \quad \text{by (a)}$$

$$\Rightarrow \dot{\underline{P}} = \underline{F}$$

$\Rightarrow$ Rate of change of momentum of the system is equal to the resultant external force on the system.

If $\underline{F} = \underline{0}$, Then

$$\dot{\underline{P}} = \underline{0}$$

$$\Rightarrow \underline{P} = \text{Constant}$$

i.e if the total external force is zero, then the total linear momentum is conserved.

<u>Note</u># Equation $M\ddot{\underline{R}}_{cm} = \underline{F}$ is valid irrespective of the points of application of external forces. However Rotational motions, as we see later, will of course be affected by the points of application of external forces.

Angular Momentum of System of Particles and its Time Rate of change

(M. Hussain Lecturer (Maths) Govt. College Asghar Mall)

We now determine the angular momentum of our general mass-system (system of particle) about the fixed point O, about the mass-centre C and about an arbitrary point which may have an acceleration $\underline{a}_p = \ddot{\underline{r}}_p$

(a) Angular Momentum about a Fixed Point.

(By Muhammad Hussain Lecturer (Maths) Govt College Asghar Mall)

The angular momentum of a system of particles about the fixed point O, fixed in the Newtonian reference system is defined as the vector sum of the moments of the linear momentums about O of all particles of the system. Hence total angular momentum (Vector) about O is.

$$\underline{L} = \sum_{i=1}^{n} (\underline{r}_i \times m\dot{\underline{r}}_i)$$

$$= \sum_{i=1}^{n} (\underline{r}_i \times m\underline{v}) \rightarrow ①$$

Differentiating w.r.t. t

$$\frac{d\underline{L}}{dt} = \sum_{i=1}^{n} (\dot{\underline{r}}_i \times m\underline{v}) + \sum_{i=1}^{n} (\underline{r}_i \times m\dot{\underline{v}})$$

$$= \underline{0} + \sum_{i=1}^{n} \underline{r}_i \times m\ddot{\underline{r}}_i$$

$$\frac{d\underline{L}}{dt} = \sum_{i=1}^{n} \underline{r}_i \times \dot{\underline{p}}_i \quad \rightarrow ② \qquad \frac{d\underline{p}_i}{dt} = \frac{d\,m\underline{v}_i}{dt}$$

This shows that the rate of change of angular momentum about the fixed point O is equal to the total moment of the rate of change of linear momentum about O.

Now

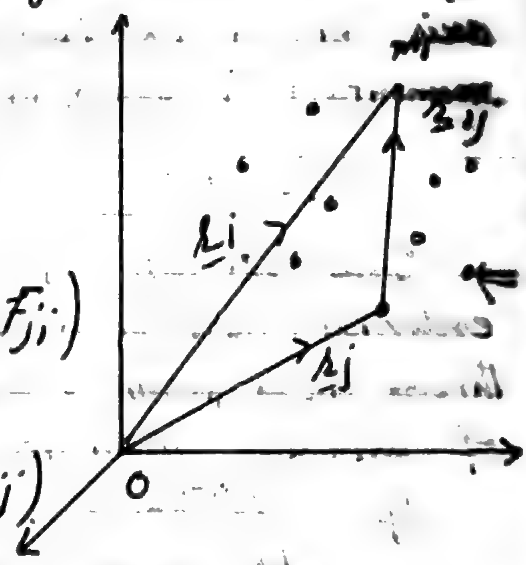
$$\dot{\underline{p}}_i = m\ddot{\underline{r}}_i = \underline{F}_i^{(e)} + \sum_{j=1}^{n} F_{ij}$$

using in ②

$$\frac{d\underline{L}}{dt} = \sum_{i=1}^{n} \underline{r}_i \times \left(\underline{F}_i^{(e)} + \sum_{j=1}^{n} F_{ij} \right)$$

$$= \sum_{i=1}^{n} \left(\underline{r}_i \times \underline{F}_i^{(e)} \right) + \sum_{i,j}^{n} \left(\underline{r}_i \times \underline{F}_{ij} \right) \rightarrow ③$$

The 2nd sum in ③ is moment or torque due to internal forces which we may denote by $\underline{N}_{int}$. This does not in general vanish but will vanish if the lines of action of all the internal forces lie along straight lines joining the particles (i.e if the internal forces are all central forces i.e follow strong law of action and reaction)



Now

$$\underline{N}_{int} = \sum_{i,j}^{n} \left(\underline{r}_i \times \underline{F}_{ij} \right)$$

$$= \sum_{i \neq j}^{n} \left(\underline{r}_i \times \underline{F}_{ij} + \underline{r}_j \times \underline{F}_{ji} \right)$$

$$= \sum_{i \neq j}^{n} \left(\underline{r}_i \times \underline{F}_{ij} - \underline{r}_j \times \underline{F}_{ij} \right)$$

$$= \sum_{i \neq j}^{n} \left(\underline{r}_i - \underline{r}_j \right) \times \underline{F}_{ij} = \sum^{n} \underline{r}_{ij} \times \underline{F}_{ij}$$

$\because$ Internal forces are central

$\therefore$ F_{ij} and r_{ij} are parallel

$$\Rightarrow \sum_{i,j}^{n} \underline{r}_i \times \underline{F}_{ij} = \sum_{ij}^{n} \underline{r}_{ij} \times \underline{F}_{ij} = \underline{0}$$

and ③ becomes

$$\frac{d\underline{L}}{dt} = \sum_{i=1}^{n} (\underline{r}_i \times \underline{F}_i^{(e)})$$

= sum of all of the external torques

$$\frac{d\underline{L}}{dt} = \underline{N}^{(e)} \longrightarrow ④$$

i-e the rate of change of vector angular momentum about a fixed point for a system of particles moving generally in space is equal to the sum of the moments of the external forces acting on the system about the point

If $\hat{a}$ is a constant unit vector along an axis through the fixed point O, about which angular momentum is taken, then from ④

$$\therefore \hat{a} \cdot \frac{d\underline{L}}{dt} = \underline{N}^{(e)} \cdot \hat{a} \longrightarrow ⑤$$

$\Rightarrow$ The resolute of the sum of moments of external forces in a fixed direction or about a line through fixed point is equal to the rate of change of angular momentum about that line

If $\underline{N}^{(e)} = \underline{0}$, Then

$$\frac{d\underline{L}}{dt} = \underline{0} \Rightarrow \underline{L} = \text{Constant}$$

$\Rightarrow$ $\underline{L}$ is constant if the applied (external)

torque is zero

from ⑤

If $\underline{N}^{(e)} \cdot \hat{a} = 0$, then $\hat{a} \cdot \frac{d\underline{L}}{dt} = 0$

$$\Rightarrow \quad \frac{d}{dt}(\underline{L} \cdot \hat{a}) = 0$$

$$\Rightarrow \quad \underline{L} \cdot \hat{a} = 0$$

⇒ If the resolute of the sum of moments of external forces in a fixed direction is zero, then the resolute (component) of the angular momentum in this direction is constant.

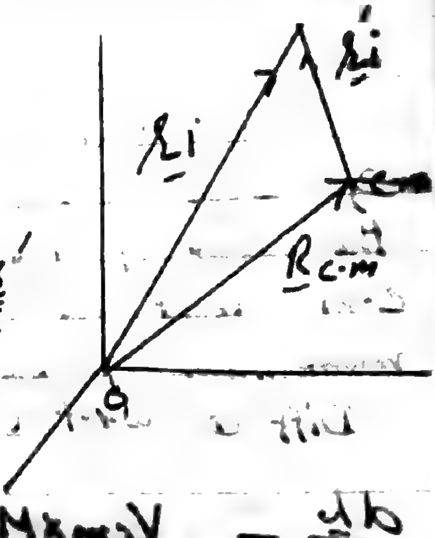
(b) <u>About Centre of Mass</u>

The angular momentum of the mass system about fixed refence point O is

$$\underline{L} = \Sigma \underline{r_i} \times \underline{p_i}$$

$$= \Sigma \underline{r_i} \times m\underline{v_i} \quad \rightarrow ⑥$$

Let $\underline{R}_{cm}$ be the radius vector from O to c·m and $\underline{r_i}'$ be the radius vector from the centre of mass C to the ith particle.



$$\underline{r_i} = \underline{r_i}' + \underline{R}_{cm}$$

and $\underline{v_i} = \underline{v_i}' + \underline{V}_{cm}$

$\underline{V}_{cm} = \frac{d\underline{R}_{cm}}{dt}$, $\underline{v_i}' = \frac{d\underline{r}'}{dt}$

using in ⑥

$$\underline{L} = \Sigma \underline{r_i} \times m(\underline{v_i}' + \underline{V}_{cm})$$

$$\underline{L} = \sum_{i=1}^{n} (\underline{r}_i' + \underline{R}_{cm}) \times m_i(\underline{v}_i' + \underline{V}_{cm})$$

$$= \sum_{i=1}^{n} \underline{r}_i' \times m_i \underline{v}_i' + \sum m_i \underline{r}_i' \times \underline{V}_{cm}$$

$$+ \sum \underline{R}_{cm} \times m_i \underline{v}_i' + \sum \underline{R}_{cm} \times m_i \underline{V}_{cm}$$

$$= \sum_{i=1}^{n} m_i \underline{r}_i' \times \underline{v}_i' + \sum m_i \underline{r}_i' \times \underline{V}_{cm}$$

$$+ \underline{R}_{cm} \times \sum m_i \underline{v}_i' + \underline{R}_{cm} \times (\sum m_i) \underline{V}_{cm}$$

$$= \sum m_i \underline{r}_i' \times \underline{v}_i' + (\sum m_i \underline{r}_i') \times \underline{V}_{cm}$$

$$+ \underline{R}_{cm} \times \frac{d}{dt}(\sum m_i \underline{r}_i') + \underline{R}_{cm} \times M \underline{V}_{cm}$$

$\because$ Sum of linear moments about c.m is zero

$\therefore$ $\sum m_i r_i' = 0$

$\Rightarrow (\sum m_i r_i') \times \underline{V}_{cm} = \underline{0}$

& $\underline{R}_{cm} \times \frac{d}{dt}(\sum m_i \underline{r}_i') = \underline{0}$

$$\underline{L} = \underline{R}_{cm} \times M\underline{V}_{cm} + \sum \underline{r}_i' \times m\underline{v}_i'$$

i.e

$$= \underline{R}_{cm} \times M\underline{V}_{cm} + \sum (\underline{r}_i' \times \underline{p}_i') \rightarrow (7)$$

$\Rightarrow$ The total angular momentum is the sum of the angular momentum of the c.m about the origin or fixed point and angular momentum of the system about the c.m.

Diff (8) w.r.t t

$$\frac{dL}{dt} = \underline{V}_{cm} \times M\underline{V}_{cm} + \underline{R}_{cm} \times M\underline{\ddot{R}}_{cm} + \sum (\underline{r}_i' \times \underline{\dot{p}}_i')$$

$$\frac{dL}{dt} = \underline{0} + \underline{R}_{cm} \times M\ddot{\underline{R}}_{cm} + \Sigma(\underline{r}_i' \times \dot{\underline{P}}_i')$$

$$= \underline{R}_{cm} \times M\ddot{\underline{R}}_{cm} + \Sigma(\underline{r}_i' \times \dot{\underline{P}}_i')$$

which shows that the rate of change of angular momentum of a system about a fixed point O is equal to the sum of the rate of change of (angular) moment of momentum of the whole mass at c.m about O and the moment of rate of change momentum of the system about centre of mass.

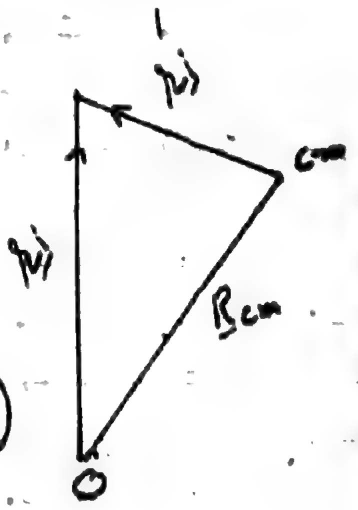
The angular momentum of the system about the mass centre C is

$$\underline{L}_c = \Sigma\, \underline{r}_i' \times m_i\dot{\underline{r}}_i \rightarrow ⑧$$

$$\underline{r}_i = \underline{r}_i' + \underline{R}_{cm}$$

$$\underline{V}_i = \underline{V}_i' + \underline{V}_{cm}$$

$$\dot{\underline{r}}_i = \dot{\underline{r}}_i' + \dot{\underline{R}}_{cm}$$



so

$$\underline{L}_c = \sum_{i=1}^{n} \underline{r}_i' \times m_i(\dot{\underline{r}}_i' + \dot{\underline{R}}_{cm})$$

$$= \sum_{i=1}^{n} \underline{r}_i' \times m_i\dot{\underline{r}}_i' + \left(\sum_{i=1}^{n} m_i \underline{r}_i'\right) \times \dot{\underline{R}}_{cm}$$

$\because \quad \Sigma\, m_i \underline{r}_i' = 0$ Sum of moments about c.m is zero

$$\underline{L}_c = \sum_{i=1}^{n} \underline{r}_i' \times m_i\dot{\underline{r}}_i' \rightarrow ⑨$$

The expression ⑧ is called absolute angular momentum because absolute velocity $\dot{\underline{r}}_i$ is used. The expression ⑨ is called the relative angular momentum about C because relative velocity $\dot{\underline{r}}_i'$ is used. We note that with the mass centre C as reference point, the absolute and relative angular momenta are identical.

Differentiating ⑧ w.r.t time

$$\dot{\underline{L}}_c = \sum \dot{\underline{r}}_i' \times m_i \dot{\underline{r}}_i + \sum \underline{r}_i' \times m_i \ddot{\underline{r}}_i$$

$$= \sum \dot{\underline{r}}_i' \times m_i (\dot{\underline{r}}_i' + \underline{R}_{cm}) + \sum \underline{r}_i' \times m_i \ddot{\underline{r}}_i$$

$$= \sum m_i (\dot{\underline{r}}_i' \times \dot{\underline{r}}_i') + \left(\sum m_i \dot{\underline{r}}_i'\right) \times \underline{R}_{cm}$$

$$+ \sum \underline{r}_i' \times m_i \ddot{\underline{r}}_i$$

$$= \underline{0} + \underline{0} + \sum \underline{r}_i' \times m_i \ddot{\underline{r}}_i$$

$$= \sum \underline{r}_i' \times m_i \ddot{\underline{r}}_i$$

$$= \sum \underline{r}_i' \times \left(\underline{F}_i^{(e)} + \sum_{j=1}^{n} \underline{F}_{ij} \right)$$

$$= \sum \underline{r}_i' \times \underline{F}_i^{(e)} + \sum_{i,j}^{n} \underline{r}_i' \times \underline{F}_{ij}$$

$$= \sum \underline{r}_i' \times \underline{F}_i^{(e)} + \underline{0}$$

= Sum of the external moments about G.M

$$\dot{\underline{L}}_c = \underline{N}_c^{(e)} \longrightarrow ⑨$$

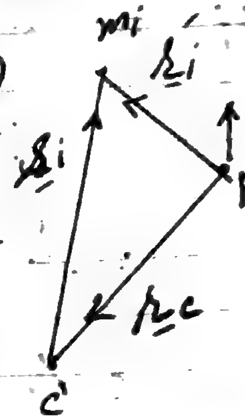
where we may use relative or the absolute angular momentum. Equation ④ & ⑨ are

among the most powerful of the governing equations in dynamics and apply to any defined system constant mass - rigid or nonrigid.

(C) About an arbitrary Point or Origin

Let C with p.v $\underline{R}_{cm}$ relative to some fixed origin be c.m and P is any point moving with velocity $\underline{v}_p = \dot{\underline{r}}_p$ and $\underline{r}_i'$ is p.v of ith particle relative to P, $\underline{\rho}_i$ is its position relative to ~~fixed origin~~ c.m. C. Then

$$\underline{L}_p = \sum \underline{r}_i' \times m_i \underline{V}_i \rightarrow (10)$$



where $\underline{V}_i = \dot{\underline{r}}_i$, $\underline{r}_i$ is p.v of m_i relative to fixed origin O.
Let $\underline{r}_c$ be p.v of C.m relative to P. Then

$$\underline{r}_i' = \underline{r}_c + \underline{\rho}_i \rightarrow (11).$$

$$\underline{v}_i = \underline{v}_p + \underline{v}_i \quad \text{where}$$

using these values in (10):

$$\underline{L}_p = \sum (\underline{r}_c + \underline{\rho}_i) \times m_i (\underline{v}_p + \dot{\underline{r}}_i')$$

$$= \sum m_i \underline{r}_c \times \underline{v}_p + \sum m_i \underline{r}_c \times \dot{\underline{\rho}}_i + \sum m_i \underline{\rho}_i \times \underline{v}_p + \sum m_i \underline{\rho}_i \times \dot{\underline{r}}_i'$$

$$= \underline{r}_c \times M \underline{v}_p + \underline{r}_c \times \frac{d}{dt}\left(\sum m_i \underline{\rho}_i\right) + \left(\sum m_i \underline{\rho}_i\right) \times \underline{v}_p + \sum \underline{\rho}_i \times m_i \dot{\underline{r}}_i'$$

$$\sum m_i \underline{\rho}_i = \underline{0}$$

Relative to P as origin

$$\underline{r}_c = \frac{\sum m_i \underline{r}_i'}{\sum m_i}$$

$$\Rightarrow \sum m_i \underline{r}_i' = M \underline{r}_c$$

Also

$$\sum \underline{s}_i \times m_i \dot{\underline{r}}_i' = \sum \underline{s}_i \times m_i (\dot{\underline{r}}_c + \dot{\underline{s}}_i)$$

$$= \sum m_i \underline{s}_i \times \dot{\underline{r}}_c + \sum \underline{s}_i \times m_i \dot{\underline{s}}_i$$

$$= \underline{0} + \sum \underline{s}_i \times m_i \dot{\underline{s}}_i$$

$$\underline{L}_P = \underline{r}_c \times M \underline{V}_P + \underline{r}_c \times M \dot{\underline{r}}_c + \sum \underline{s}_i \times m_i \dot{\underline{s}}_i$$

$$= \underline{r}_c \times M (\underline{V}_P + \dot{\underline{r}}_c) + \sum \underline{s}_i \times m_i \dot{\underline{s}}_i$$

$$= \underline{r}_c \times M \underline{V}_{cm} + \sum \underline{s}_i \times m_i \dot{\underline{s}}_i \rightarrow (12)$$

where $\underline{V}_{cm} = \underline{V}_P + \dot{\underline{r}}_c$ is velocity of centroid relative to fixed origin O

$$\underline{L}_P = \sum \underline{s}_i \times m_i \dot{\underline{s}}_i + \underline{r}_c \times M \underline{V}_{cm}$$

$$= \underline{L}_{cm} + \underline{r}_c \times M \underline{V}_{cm} \rightarrow (13)$$

It states that absolute angular momentum of the system of particles about any point P is equal to the angular momentum about C.M. plus the moment about P of single particle of total mass equal to that of the entire system concentrated at its centroid and moving with the centroid's velocity.

Diff (13) w.r.t t

$$\dot{\underline{L}}_p = \dot{\underline{L}}_{cm} + \dot{\underline{r}}_c \times M\underline{V}_{cm} + \underline{r}_c \times M\dot{\underline{V}}_{cm}$$

But $\underline{V}_{cm} = \underline{V}_p + \dot{\underline{r}}_c$

$$\Rightarrow \dot{\underline{r}}_c = \underline{V}_{cm} - \underline{V}_p$$

$$\Rightarrow \dot{\underline{L}}_p = \dot{\underline{L}}_{cm} + (\underline{V}_{cm} - \underline{V}_p) \times M\underline{V}_{cm} + \underline{r}_c \times M\dot{\underline{V}}_{cm}$$

$$= \dot{\underline{L}}_{cm} + \underline{V}_{cm} \times M\underline{V}_{cm} - \underline{V}_p \times M\underline{V}_{cm} + \underline{r}_c \times M\dot{\underline{V}}_{cm}$$

$$= \dot{\underline{L}}_{cm} + \underline{r}_c \times M\dot{\underline{V}}_{cm} - \underline{V}_p \times M\underline{V}_{cm} \rightarrow (14)$$

Let $P \equiv C$, Then $\underline{r}_c = \underline{0}$

and from

$$\dot{\underline{r}}_c = \underline{V}_{cm} - \underline{V}_p$$

we have

$$\underline{V}_p = \underline{V}_{cm}$$

$$\dot{\underline{L}}_p = \dot{\underline{L}}_{cm} = \frac{d}{dt}\left(\Sigma \underline{s}_i \times m_i \dot{\underline{s}}_i\right) = \left(\Sigma \underline{s}_i \times m \ddot{\underline{s}}_i\right) \rightarrow (15)$$

i.e the rate of change of angular momentum of the system of particulars about its centroid is equal to the total moment of the rate of change of momentum about centroid of the system in its motion relative to centroid.

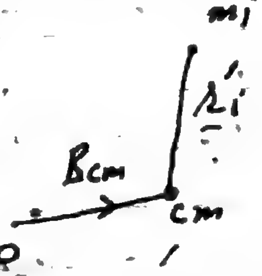
In (15), we have calculated rate of change of momenumt about C.m when C.m is moving.

Now rate of change of angular momentum about a fixed point O is

$$\frac{d\underline{L}}{dt} = \underline{R}_{cm} \times M\ddot{\underline{R}}_{cm} + \Sigma\, \underline{r}_i' \times \dot{\underline{P}}_i')$$

$$= \underline{R}_{cm} \times M\ddot{\underline{R}}_{cm} + \Sigma\, \underline{r}_i' \times m_i \dot{\underline{v}}_i'$$

If we take $\underline{R}_{cm} = \underline{0}$ i.e we consider c.m at fixed point or if we consider c.m as fixed point, then $\underline{R}_{cm} = \underline{0}$ and



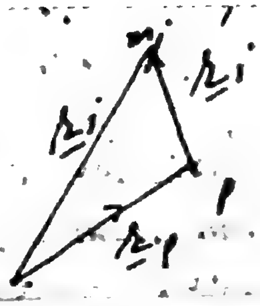
$$\frac{d\underline{L}}{dt} = \Sigma\, \underline{r}_i' \times m_i \dot{\underline{v}}_i' = \Sigma\, \underline{r}_i' \times m \ddot{\underline{r}}_i' \quad \rightarrow (16)$$

Here $\underline{L}$ is now about c.m.

By comparing (15) and (16) we note that when calculating the rate of change of angular momentum of the particle system about its centroid we may treat the centroid as if it were at rest.

Theorem# Prove that the rate of change . angular momentum of a system particles about its centroid is always equal to the vector sum of the moments about the centroid, irrespective centroid is moving or at rest

Proof# Let O be fixed origin and let $\underline{F}_i^{(e)}$ be total external force on the particle at P and let P be point moving w.r.t O. Let $\underline{L}_o$, $\underline{L}_p$ angular momenta about O & P; $\underline{N}_o$, $\underline{N}_p$ torques about O and



$$\underline{N}_o = \sum \underline{r}_i \times \underline{F}_i^{(e)}$$

$$= \sum (\underline{r}_p + \underline{r}_i') \times \underline{F}_i^{(e)}$$

$$= \underline{r}_p \times \sum \underline{F}_i^{(e)} + \sum \underline{r}_i' \times \underline{F}_i^{(e)} \qquad \underline{r}_i = \underline{r}_p + \underline{r}_i'$$

$$\underline{N}_o = \underline{r}_p \times \sum \underline{F}_i^{(e)} + \underline{N}_p$$

$$\dot{\underline{L}}_o = \sum (\underline{r}_p + \underline{r}_i') \times \underline{F}_i^{(e)} = \underline{r}_p \times \sum \underline{F}_i^{(e)} + \underline{N}_p$$

$$\dot{\underline{L}}_o = \underline{r}_p \times \sum \underline{F}_i^{(e)} + \underline{N}_p \rightarrow (1)$$

But

$$\underline{L}_o = \sum (\underline{r}_p + \underline{r}_i') \times m_i \underline{v}_i$$

$$= \sum \underline{r}_p \times m_i \underline{v}_i + \sum \underline{r}_i' \times m_i \underline{v}_i$$

$$= \underline{r}_p \times \sum m_i \underline{v}_i + \underline{L}_p$$

$$= \underline{r}_p \times M \underline{V}_{cm} + \underline{L}_p \rightarrow (2)$$

Diff (2) w.r.t t

$$\dot{\underline{L}}_o = \dot{\underline{r}}_p \times M \underline{V}_{cm} + \underline{r}_p \times M \dot{\underline{V}}_{cm} + \dot{\underline{L}}_p \rightarrow (3)$$

By (1) & (3)

$$\underline{r}_p \times \sum \underline{F}_i^{(e)} + \underline{N}_p = \dot{\underline{r}}_p \times M \underline{V}_{cm} + \underline{r}_p \times M \dot{\underline{V}}_{cm} + \dot{\underline{L}}_p$$

$$\because \sum \underline{F}_i^{(e)} = M \dot{\underline{V}}_{cm}$$

$$\Rightarrow \underline{r}_p \times M \dot{\underline{V}}_{cm} + \underline{N}_p = \dot{\underline{r}}_p \times M \underline{V}_{cm} + \underline{r}_p \times M \dot{\underline{V}}_{cm} + \dot{\underline{L}}_p$$

$$\dot{\underline{L}}_p = \underline{N}_p - \dot{\underline{r}}_p \times M \underline{V}_{cm} \rightarrow (4)$$

Let $P \equiv C$ (Centroid) then $\underline{r}_c = \underline{0}$

$\therefore \dot{L}_p = \dot{L}_c = N_c$

Thus the rate of change of angular momentum of particle system about centroid is always equal, to vector sum of the moments about the centroid of all the external forces, irrespective of whether centroid be moving or at rest. (proved)

If F_{ij} is internal force on particle then.

$$m_i \dot{v}_i = F_i^{(e)} + \sum f_{ij}$$

$$\therefore \sum r' \times m_i \dot{v}_i = \sum r' \times F_i^{(e)} + \sum_{i,j}^{n} r'_i \times F_{ij}$$

$$= \sum r'_i \times F_i^{(e)} + 0 \rightarrow ⑥$$

i.e the total moment of the rate of change of momentum of the system about any point moving, or fixed, is always equal to the total moment of the external forces about that point.

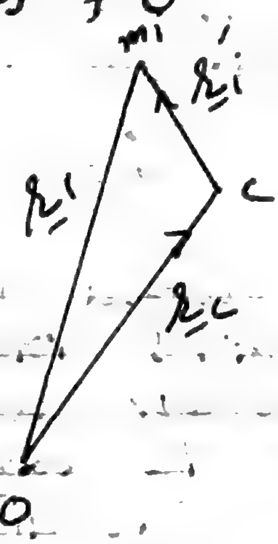
When P is fixed or coincident with centroid, the R.H.S of ⑤ has been shown equal to rate of change of moment of momentum about the point is always equal to the total moment of the external forces about the point.

Again

$\cancel{m_i \dot{v}_i} =$

Theorem# Prove that the angular momentum of the system about O is equal to the sum of the angular momenta about O of motion relative to O and that ~~of a~~ particle of mass $M = \sum m_i$ at

Centroid C moving with velocity of O



Proof# Angular momentum about O is

$$\underline{L}_0 = \sum \underline{r}_i \times m_i \underline{V}_i$$

Let $\underline{V}_0$ be velocity of O

$$\underline{L}_0 = \sum \underline{r}_i \times m_i(\underline{V}_i - \underline{V}_0 + \underline{V}_0)$$

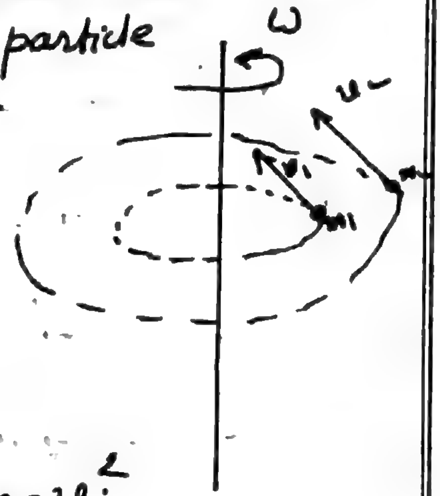
$$= \sum \underline{r}_i \times m_i(\underline{V}_i - \underline{V}_0) + \sum \underline{r}_i \times m_i \underline{V}_0$$

$$= \sum \underline{r}_i \times m_i(\underline{V}_i - \underline{V}_0) + \underline{r}_c \times M \underline{V}_0$$

= Angular momentum about O due to motion relative to O + Momentum of particle of mass $\sum m_i = M$ at C moving with velocity $\underline{V}_0$ of O

K.E and Angular Momentum of a Rigid Body Rotating about fixed Axis

Suppose a rigid body consisting of n particles rotates about a fixed axes with angular velocity ω. Each particle of the body will describe a circle around the axis of rotation with angular speed ω. Velocity v_i of ith particle is given by



$$v_i = \omega r_i$$

$$K.E \text{ of ith particle} = \frac{1}{2} m v_i^2$$

$$= \frac{1}{2} m r_i^2 \omega^2$$

total K.E of rigid by

$$K.E = \sum_{i=1}^{n} \frac{1}{2} m_i r_i^2 \omega^2$$

$$= \left(\sum_{i=1}^{n} \frac{1}{2} m_i r_i^2\right) \omega^2$$

$$= \frac{1}{2} \left(\sum_{i=1}^{n} m_i r_i^2\right) \omega^2$$

But $I = \sum_{i=1}^{n} m_i r_i^2$ moment of inertial of the body

$$K.E = \frac{1}{2} I \omega^2 \longrightarrow ①$$

This is analogous to the expression for

translation K.E. $\frac{1}{2}mv^2$

Angular Momentum of ith. particle is

$$\underline{L}_i = \underline{R}_i \times m_i \underline{v}_i$$ where R_i is P.V. from some fixed point on axis

$$|\underline{L}_i| = |\underline{R}_i \times m_i \underline{v}_i|$$

$$= m_i R_i v_i \sin\theta$$

$$= m_i r_i v_i \qquad R_i \sin\theta = r_i$$

But $v_i = \omega r_i$

$$L_i = m_i r_i^2 \omega$$

total angular momentum about the axis of rotation.

$$L = \sum m_i r_i^2 \omega$$

$$L = I\omega$$

In vector form

$$\underline{L} = I\underline{\omega} \longrightarrow (2)$$

In this case $\underline{L}$ is parallel to angular velocity. This is not true for general motion of rigid body.

If N is total external moment of the external forces about the fixed axis, then

$$\frac{dL}{dt} = N$$

$$\frac{d}{dt}(I\omega) = N$$

$$I\dot{\omega} = N$$

$$I\alpha = N \quad \text{where } \alpha = \frac{d\omega}{dt}$$
is angular acceleration.

The work done during infinitesimal rotation $d\theta$ is given by

$$dW_{net} = N\,d\theta$$

$$dW_{net} = N\,\omega dt$$

$$\Rightarrow \frac{dW_{net}}{dt} = N\omega$$

$$P = N\omega$$

which gives the instantaneous mechanical power P

Splitting up of K.E of Rigid Body into K.E of Translation and K.E of Rotation#

Problem Prove that the K.E of a rigid body can be separated into two parts, one associated with the pure translation of centre of mass of the body and the other associated with pure rotation about an axis through the centre of mass. Also write K.E in tensorial form

Sol# Consider a rigid body composed of n particles of masses m_α, $\alpha = 1, 2, \ldots, n$. If this body rotates with an instantaneous angular velocity ω about some point fixed w.r.t

the body co-ordinate system, and if this point moves with velocity $\underline{V}$ (instantaneous) w.r.t to the fixed co-ordinate system, then the instantaneous velocity of αth particle in the fixed system is given by

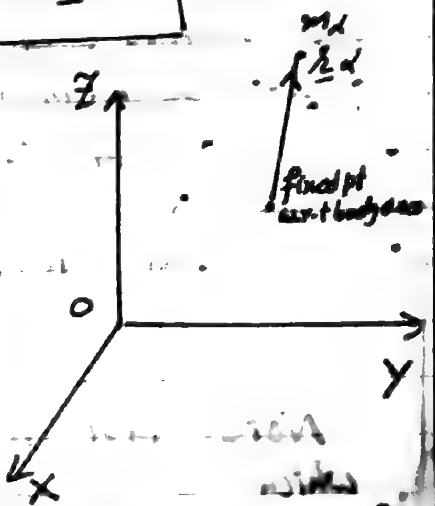
$$\underline{v}_\alpha = \underline{V} + \underline{v}_r + \underline{\omega} \times \underline{r}_\alpha$$

Since body is rigid

$$\underline{v}_r = \left(\frac{dr}{dt}\right)_{rotating} = \underline{0}$$

Therefore

$$\boxed{\underline{v}_\alpha = \underline{V} + \underline{\omega} \times \underline{r}_\alpha} \longrightarrow (1)$$



where the subscript f, for fixed co-ordinate system is dropped from velocity v_α. It is now understood that all velocities are measured in the fixed system; all velocities w.r.t to the rotating or body system vanish because the body is rigid. Relative to O i.e absolute K.E of the αth particle is given by

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2 \longrightarrow (2)$$

and total K.E of the body is

$$T = \frac{1}{2} \sum_\alpha m_\alpha (\underline{V} + \underline{\omega} \times \underline{r}_\alpha)^2$$

$$= \frac{1}{2} \sum_\alpha m_\alpha \left[(\underline{V} + \underline{\omega} \times \underline{r}_\alpha) \cdot (\underline{V} + \underline{\omega} \times \underline{r}_\alpha)\right]$$

$$= \frac{1}{2} \sum_\alpha m_\alpha \left[\underline{V} \cdot \underline{V} + \underline{V} \cdot \underline{\omega} \times \underline{r}_\alpha + (\underline{\omega} \times \underline{r}_\alpha) \cdot \underline{V} + (\underline{\omega} \times$$

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left[V^2 + 2\,\underline{V}\cdot\underline{\omega}\times\underline{r}_\alpha + (\underline{\omega}\times\underline{r}_\alpha)^2 \right]$$

$$= \frac{1}{2}\sum_\alpha m_\alpha V^2 + \sum m_\alpha\, \underline{V}\cdot\underline{\omega}\times\underline{r}_\alpha + \frac{1}{2}\sum_\alpha m_\alpha(\underline{\omega}\times\underline{r}_\alpha)^2 \quad \longrightarrow (3)$$

This is general expression for the K·E and is valid for any choice of the origin from which vectors $\underline{r}_\alpha$ are measured.

If we take the origin of the body co-ordinate system coincident with centre of mass, then $\underline{r}_\alpha$ will be measured from c·m. In equation (3) neither $\underline{V}$ nor $\underline{\omega}$ is characteristic of the αth particle and therefore they may be taken out of the summation.

$$\sum_\alpha m_\alpha\, \underline{V}\cdot\underline{\omega}\times\underline{r}_\alpha = \underline{V}\cdot\underline{\omega}\times\left(\sum_\alpha m_\alpha \underline{r}_\alpha\right) \quad \longrightarrow (4)$$

But $\underline{R}$ (p.v. of c·m) $= \dfrac{\sum_\alpha m_\alpha \underline{r}_\alpha}{\sum_\alpha m_\alpha = M}$

$$\Rightarrow \quad \sum_\alpha m_\alpha \underline{r}_\alpha = M\underline{R}$$

Note that $\underline{R}$ is independent of origin from which $\underline{r}_\alpha$ is measured. But when $\underline{r}_\alpha$ is measured from c·m, then $\underline{R} = \underline{0}$ & hence $\sum_\alpha m_\alpha \underline{r}_\alpha = \underline{0}$

So from (4)

$$\sum_\alpha m_\alpha\, \underline{V}\cdot\underline{\omega}\times\underline{r}_\alpha = \underline{0}$$

Thus K·E can be written as

$$T = \frac{1}{2}\sum_\alpha m_\alpha V^2 + \frac{1}{2}\sum_\alpha m_\alpha(\underline{\omega}\times\underline{r}_\alpha)^2$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}\sum_\alpha m_\alpha(\underline{\omega}\times\underline{r}_\alpha)^2$$

$$= T_{trans} + T_{rot}$$

where

$$T_{trans} = \frac{1}{2}\sum_{\alpha} m_\alpha V^2 = \frac{1}{2}MV^2 \rightarrow (5)$$

is translational K.E

and $$T_{rot} = \frac{1}{2}\sum m_\alpha (\omega \times r_\alpha)^2 \rightarrow (6)$$

is rotational K.E due to ratation of the system about an axis through the c.m.

Tensorial Notation

The rotational K.E can be expanded by using formula

$$(A \times B)^2 = (A \times B)\cdot(A \times B)$$

$$= \begin{vmatrix} A\cdot A & A\cdot B \\ B\cdot A & B\cdot B \end{vmatrix}$$

$$= A^2B^2 - (A\cdot B)^2$$

$$T_{rot} = \frac{1}{2}\sum_{\alpha} m_\alpha \left[\omega^2 r_\alpha^2 - (\omega\cdot r_\alpha)^2\right] \rightarrow (7)$$

We now express T_{rot} by making use of the components ω_i, $r_{\alpha,i}$ of the vectors ω and r_α. We also note that

$$r_\alpha = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$$

in the body co-ordinates system x_i, ... axes. So we can write

$$r_{\alpha,i} = x_{\alpha,i}$$

Then

$$T_{rot} = \frac{1}{2}\sum_{\alpha} m_\alpha \left[\left(\sum_i \omega_i^2\right)\left(\sum_{k=1}^{3} x_{\alpha,k}^2\right) - \left(\sum_i \omega_i x_{\alpha,i}\right)\left(\sum_j \ldots\right)\right]$$

Now $$\omega_i = \sum_j \omega_j \delta_{ij}$$

$$T_{rot} = \frac{1}{2}\sum_{\alpha}\sum_{i,j} m_\alpha\left[\omega_i\omega_j\delta_{ij}\left(\sum_k x_{\alpha,k}^2\right) - \omega_i\omega_j x_{\alpha,i}\,x_{\alpha,j}\right]$$

$$= \frac{1}{2}\sum_{i,j}\omega_i\omega_j \sum_\alpha m_\alpha\left[\delta_{ij}\left(\sum_k x_{\alpha,k}^2\right) - x_{\alpha,i}\,x_{\alpha,j}\right] \longrightarrow (9)$$

If we define the ijth element of the sum over α to I_{ij}, then

$$I_{ij} = \sum_\alpha m_\alpha\left[\delta_{ij}\sum_k x_{\alpha;k}^2 - x_{\alpha,i}\,x_{\alpha,j}\right] \longrightarrow (10)$$

and

$$\boxed{T_{rot} = \frac{1}{2}\sum_{i,j} I_{ij}\,\omega_i\omega_j} \longrightarrow (11)$$

If we consider a body as a continuous distribution of matter with mass density $\rho = \rho(\underline{r})$

Then

$$I_{ij} = \int_V \rho(\underline{r})\left[\delta_{ij}\sum x_k^2 - x_i x_j\right]dV$$

where $dV = dx_1 dx_2 dx_3$ is the volume element at the position vector $\underline{r}$ and V is the volume of the body

Remarks — T_{trans} & T_{rot} are quite independent ; the rotation would be present even in the absence of translation (e.g. as observed from a frame of reference moving with $\underline{V}$ i.e velocity of c.m because an observer viewing the system from an inertial frame moving with $\underline{V}$ will see the c.m standing still. To this observer the basic equation of rotational dynamics $\sum \tau = I\alpha$ will still apply provided (1) the axis of rotation passes through c.m (2) the axis always has same direction in space i.e as the system moves, the axis at one instant is parallel to to the axis at any other instant.

K·E of a Rigid Body in General Motion.

Problem# Prove that K·E of rigid body in general motion is given by

$$K\cdot E = T = T_{rot} + T_{trans} + T_m$$

where T_{rot} is K·E due to rotation

T_{tran} = K·E due to translation

T_m = mixed energy which is determined by translation and the rotation combined.

Sol# Consider a rigid body composed of n particles of masses m_i $i = 1, 2, \ldots n$. If this body rotates with an instantaneous angular velocity $\underline{\omega}$ about some pt. fixed w.r.t body co-ordinate system. Let $\underline{V}$ be velocity of translation of body. Then the instantaneous velocity of ith particle of the body w.r.t fixed system is

$$\underline{v}_i = \underline{V} + \underline{\omega} \times \underline{r}_i \longrightarrow (1)$$

where $\underline{r}_i$ is p.v of ith particle from point of body about which rotation is considered. The choice of this reference point is upto us. For many purposes it is useful to take this reference point at the mass centre. For reference point $\underline{v}_i = \underline{V}$

$\therefore$ K.E of ith particle relative fixed Co-ordinate system or inertial system is

$$T_i = \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} m_i \left(\underline{V} + \underline{\omega} \times \underline{r}_i \right)^2$$

Total K.E of the body is

$$T = \sum_{i=1}^{n} T_i = \frac{1}{2} \sum m_i \left(\underline{V} + \underline{\omega} \times \underline{r}_i \right)^2$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i \left[\left(\underline{V} + \underline{\omega} \times \underline{r}_i \right) \cdot \left(\underline{V} + \underline{\omega} \times \underline{r}_i \right) \right]$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i \left[\underline{V} \cdot \underline{V} + \underline{V} \cdot \underline{\omega} \times \underline{r}_i + \underline{\omega} \times \underline{r}_i \cdot V + (\underline{\omega} \times \underline{r}_i)^2 \right]$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i \left[V^2 + 2 \underline{V} \cdot \underline{\omega} \times \underline{r}_i + (\underline{\omega} \times \underline{r}_i)^2 \right]$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i V^2 + \sum_{i=1}^{n} m_i (\underline{V} \cdot \underline{\omega} \times \underline{r}_i) + \sum \frac{1}{2} m_i (\underline{\omega} \times \underline{r}_i)^2$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} \sum m_i (\underline{\omega} \times \underline{r}_i)^2 + \sum_{i=1}^{n} m_i (\underline{V} \cdot \underline{\omega} \times \underline{r}_i) \quad (1)$$

$$= T_{trans} + T_{rot} + T_m$$

where $T_m = \sum_{i=1}^{n} m_i \underline{V} \cdot \underline{\omega} \times \underline{r}_i$

$$= \underline{V} \cdot \underline{\omega} \times \sum m_i \underline{r}_i$$

$$= \underline{V} \cdot \underline{\omega} \times M \underline{R}_{cm}$$

where $\underline{R}_{cm} = \dfrac{\sum_{i=1}^{n} m_i \underline{r}_i}{M}$

is P.V of c.m and is independent of the reference point from where $\underline{r}_i$ are measured. But w.r.t c.m $\underline{R}_{cm} = \underline{0}$

$T_m = M \underline{V} \cdot \underline{\omega} \times \underline{R}_{cm}$ is mixed energy determined by translation and the rotation combined.

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Angular Momentum of Rigid Body

With respect to some point O that is fixed in the body co-ordinate system, the angular momentum of the body is

$$\underline{L} = \sum_{\alpha=1}^{n} \underline{r}_\alpha \times \underline{P}_\alpha$$

$$= \sum_{\alpha=1}^{n} \underline{r}_\alpha \times m_\alpha \underline{v}_\alpha$$

$$= \sum_{\alpha=1}^{n} m_\alpha (\underline{r}_\alpha \times \underline{v}_\alpha)$$

The most convenient choice for the position of the point O depends upon the particular problem. There are only two choices of ... (a) if one or more points of the body are fixed (in the fixed co-ordinate system), O is ... to coincide with one such point (b) if no ... of the body is fixed, O is chosen to be ... centre of mass.

We will discuss the following cases

(1)# Angular momentum of rigid body about a

stationary point of body

(2) Angular momentum about C.m of body

(3) Angular momentum of a rigid body about a stationary point of body in terms of its angular momentum of the body about its C.m

(4) We will also prove that when a body rotates about a fixed point or when angular momentum is about C.m of body moving with general motion, we have

$$[L] = [I][\omega]$$

where $[I]$ = Interia matrix

$[L]$ = Angular momentum Matrix

$[\omega]$ = Angular velocity matrix

(5)# We shall also prove that angular momentum of body relative to a point fixed in body co-ordinate system is given by

$$\underline{h} = \{I\} \cdot \underline{\omega} = \underline{I} \cdot \underline{\omega}$$

where $\{I\} = \underline{I}$ is inertia tensor and ... is dot produt of a tensor with a ...

... shall prove the relation

$$T_{rot} = \frac{1}{2}\, \underline{\omega} \cdot \underline{h} = \frac{1}{2}(\underline{\omega} \cdot \{I\} \cdot \underline{\omega})$$

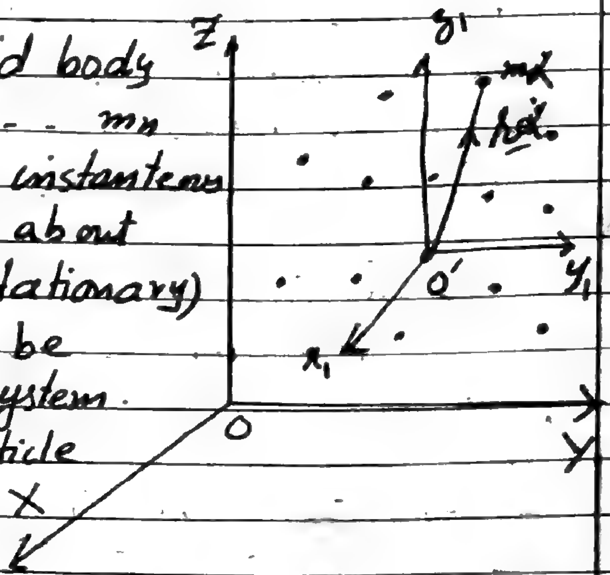
... $\underline{h}$ is angular momentum of body about a point fixed in body Co-ordinates system and ... is instantaneous angular velocity about the point

Angular Momentum in Tensorial Form

Problem*# Define angular momentum of a rigid body and express it in tensor form #

Sol#

Consider a rigid body of masses $m_1, m_2, \ldots, m_n$ which rotatates with instantenus angular velocity ω about same fixed point (stationary) point O'. Let $O'x_1y_1z_1$ be body Co-ordinate system. Velocity v_α of αth particle relative to fixed system $OXYZ$ is given by



$$v_\alpha = \omega \times r_\alpha \quad \rightarrow (1)$$

Relative to body Co-ordinate system or relative to O' linear momentum of αth particle is

$$\underline{P_\alpha} = m_\alpha \underline{v_\alpha} = m_\alpha \underline{\omega} \times \underline{r_\alpha} \rightarrow (2)$$

Hence angular momentum of body relative to O' (stationary point) is

$$\underline{h} = \sum_{\alpha=1}^{n} \underline{r_\alpha} \times \underline{p_\alpha}$$

$$= \sum_{\alpha=1}^{n} \underline{r_\alpha} \times m_\alpha \underline{\omega} \times \underline{r_\alpha}$$

$$= \sum m_\alpha \, \underline{r_\alpha} \times (\underline{\omega} \times \underline{r_\alpha})$$ where

$$\underline{L} = \sum m_\alpha \underline{r}_\alpha \times (\underline{\omega} \times \underline{r}_\alpha)$$

$$= \sum_{\alpha=1}^{n} m_\alpha \left[r_\alpha^2 \underline{\omega} - \underline{r}_\alpha (\underline{r}_\alpha \cdot \underline{\omega}) \right] \rightarrow (3)$$

To express in tensorial form. Let $\omega_i, r_{\alpha,i}$ be components of $\underline{\omega}, \underline{r}_\alpha$ relative to $O'x_1y_1z_1$ and L_i $i=1,2,3$ be components of vector $\underline{L}$.

Now $\underline{r}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$ and so

$$r_{\alpha,i} = x_{\alpha,i}$$

For ith component of $\underline{L}$, we have from (3)

$$L_i = \sum_\alpha m_\alpha \left[\omega_i \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} \sum_{j=1}^{3} x_{\alpha,j} \omega_j \right]$$

$$\omega_i = \sum_j \delta_{ij} \omega_j$$

$$L_i = \sum_\alpha m_\alpha \left[\omega_j \delta_{ij} \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} \sum x_{\alpha,j} \omega_j \right]$$

$$= \sum_\alpha m_\alpha \sum_j \left[\omega_j \delta_{ij} \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \omega_j \right]$$

$$= \sum_j \omega_j \sum_\alpha m_\alpha \left[\delta_{ij} \sum x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right]$$

$$= \sum_j I_{ij} \omega_j$$

where $I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right]$

$= (i,j)$th element of inertia tensor.

$$\boxed{L_i = \sum_{j=1}^{3} I_{ij}\,\omega_j} \longrightarrow (4)$$

From (4) we note that the inertia tensor $\{I\}$ relates a sum over the components of the angular velocity vector to the ith component of the angular momentum vector.

In terms of dot product of dyadic tensor (tensor in matrix form, where matrix is nonion form of a dyadic) equation (4) can be written as

$$\underline{L} = \{I\} \cdot \underline{\omega}$$

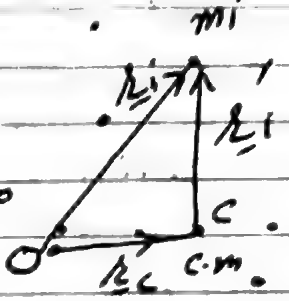
which shows that product of a tensor and a vector gives a vector.

Angular Momentum of a Rigid Body about a stationary point of body in terms angular Momentum of body about its C.m

Theorem# Prove that the angular momentum of rigid body about a stationary point is equal to the angular momentum about C.m plus the angular momentum about the stationary point due to translation of c.m relative to the point.

Proof# Consider a rigid body consisting of masses $m_1, m_2, \dots, m_n$ rotating with instantaneous angular velocity $\underline{\omega}$ about

a stationary point O. Let $\underline{r}_c$ p.v of c.m w.r.t O, $\underline{r}_i$ be p.v of ith particle w.r.t O and $\underline{r}_i'$ be p.v of ith particle w.r.t centre of mass. Then



$$\underline{r}_i = \underline{r}_c + \underline{r}_i' \rightarrow (1)$$

Angular momentum of body about O is given by

$$\underline{L}_0 = \sum_{i=1}^{n} m_i \underline{r}_i \times (\underline{\omega} \times \underline{r}_i)$$

Putting $\underline{r}_i = \underline{r}_c + \underline{r}_i'$

$$\underline{L}_0 = \sum_{i=1}^{n} \left[m_i (\underline{r}_c + \underline{r}_i') \times \{ \underline{\omega} \times (\underline{r}_c + \underline{r}_i') \} \right]$$

$$\sum_{i=1}^{n} \left[m_i (\underline{r}_c + \underline{r}_i') \times (\underline{\omega} \times \underline{r}_c + \underline{\omega} \times \underline{r}_i') \right]$$

$$= \sum^{n} m_i \left[\underline{r}_c \times (\underline{\omega} \times \underline{r}_c) + \underline{r}_c \times (\underline{\omega} \times \underline{r}_i') + \underline{r}_i' \times (\underline{\omega} \times \underline{r}_c) + \underline{r}_i' \times (\underline{\omega} \times \underline{r}_i') \right]$$

$$= \Sigma m_i \underline{r}_c \times (\underline{\omega} \times \underline{r}_c) + \underline{r}_c \times (\underline{\omega} \times \Sigma m_i \underline{r}_i')$$

$$+ \sum_{i=1}^{n} m_i \underline{r}_i' \times (\underline{\omega} \times \underline{r}_c) + \Sigma m_i \underline{r}_i' \times (\underline{\omega} \times \underline{r}_i')$$

But $\Sigma m_i = M =$ total mass of body

$\Sigma m_i \underline{r}_i' =$ sum of linear moments about c.m $= 0$

$\sum m_i \underline{r}_i \times (\underline{\omega} \times \underline{r}_i)$ = angular momentum of the rigid body about c.m = $\underline{L}_{cm}$

$\sum m_i \underline{r}_c \times (\underline{\omega} \times \underline{r}_c) = M \underline{r}_c \times (\underline{\omega} \times \underline{r}_c)$

is angular momentum about o due to translation of c.m relative to o

Thus

$$\underline{L}_O = M \underline{r}_c \times (\underline{\omega} \times \underline{r}_c) + \underline{L}_{c.m}$$

$\Rightarrow$ Angular momentum about a stationary point o is equal to the angular momentum about c.m plus angular momentum about o due to the translation of c.m

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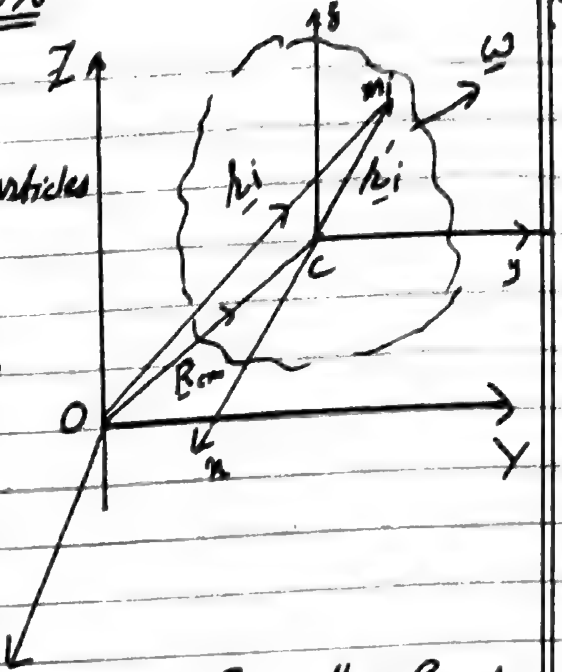
Similarity Between the Expressions For angular Momentum about a stationary Point and angular momentum about c.m of Body in General Motion

Problem # Prove that the angular momentum of a rigid body a stationary point and angular momentum of the rigid body in general motion about c.m have identical forms.



Sol# Consider a rigid body consisting of n-particles moving with general motion in space. Axes x-y-z are attached to the body with origin at centre of mass C.
The angular velocity of the body becomes the angular velocity of x-y-z frame as observed from the fixed reference axes X-Y-Z. Let $\underline{r}_i$ be p.v of ith particle w.r.t O and $\underline{r}_i'$ be its p.v w.r.t c.m, C. If $\underline{R}_{cm}$ is p.v of C w.r.t O. Then,

$$\underline{r}_i = R_{cm} + \underline{r}_i' \longrightarrow (1)$$

$$\underline{v}_i = \underline{V}_{cm} + \underline{v}_i'$$

$$\underline{v}_i = \underline{V}_{cm} + \underline{\omega} \times \underline{r}_i'$$

where $\underline{\omega}$ is instantaneous angular velocity and $\underline{v}_i' = \underline{\omega} \times \underline{r}_i'$ is the relative ~~angula~~ velocity of m_i w.r.t C.

The absolute angular momentum $\underline{L}_{cm}$ about its c.m, C is given by

$$\underline{L}_{cm} = \sum_{i=1}^{n} \underline{r}_i' \times m_i \underline{v}_i$$

$$= \sum_{i=1}^{n} \underline{r}_i' \times m_i (\underline{V}_{cm} + \underline{\omega} \times \underline{r}_i')$$

$$= \sum_{i=1}^{n} \underline{r}_i' \times m_i \underline{V}_{cm} + \sum_{i=1}^{n} \underline{r}_i' \times m_i (\underline{\omega} \times \underline{r}_i')$$

$$\underline{L}_{cm} = \sum_{i=1}^{n} \underline{r}_i' m_i \times \underline{V}_{cm} + \sum_{i=1}^{n} \underline{r}_i' \times m_i (\underline{\omega} \times \underline{r}_i')$$

$$\because \sum_{i=1}^{n} \underline{r}_i' m_i = \underline{0}$$

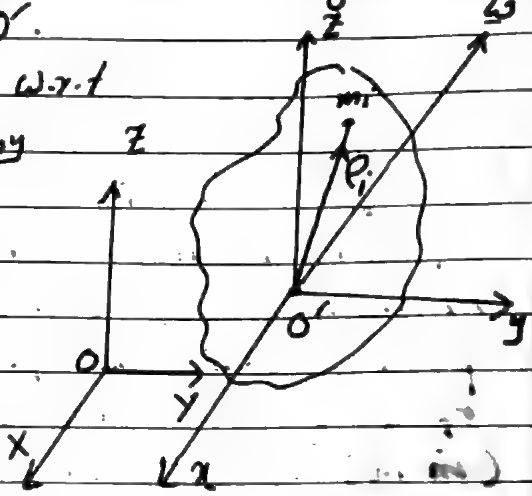
$$\underline{L}_{cm} = \sum_{i=1}^{n} \underline{r}_i' \times m_i (\underline{\omega} \times \underline{r}_i') \rightarrow ②$$

In case of continuous distribution of mass

$$\underline{L}_{cm} = \int \underline{r}' \times (\underline{\omega} \times \underline{r}') dm \rightarrow ③$$

Again considering the motion of rigid body about a fixed point O'.

Let $\underline{\rho}_i$ be p.v of m_i w.r.t O'. Then $\underline{v}_i$ is given by



$$\underline{v}_i = \underline{v}_{O'} + \underline{\omega} \times \underline{\rho}_i$$

$$\underline{v}_i = \underline{0} + \underline{\omega} \times \underline{\rho}_i$$

Angular momentum of body about O' is

$$\underline{L}_{O'} = \sum \underline{\rho}_i \times m_i \underline{v}_i$$

$$= \sum_{i=1}^{n} \underline{\rho}_i \times m_i (\underline{\omega} \times \underline{\rho}_i)$$

In case of continuous distribution of mass

$$\underline{L}_{O'} = \int \underline{\rho}_i \times (\underline{\omega} \times \underline{\rho}_i) dm$$

$$= \int \underline{\rho} \times (\underline{\omega} \times \underline{\rho}) dm \rightarrow ④$$

From ② & ④ we note that angular momentum

about stationary point and angular momentum about c.m in case of general motion are identical in form. Also it comes out that angular momentum about c.m whether c.m is stationary or traslating is same.

---

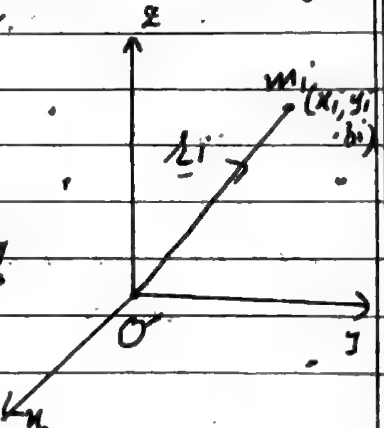
*Proof of $[\underline{L}] = [I][\underline{\omega}]$

Problem # Derive the relation

$$[\underline{L}] = [I][\underline{\omega}]$$

where $\underline{L}, I, \underline{\omega}$ have their usual meaning

Proof # Consider a rigid body consisting of n particles. The angular momentum of body about a poin O' which be fixed point of body or O' may be c.m (in this case O' may be translating or stationary) is given by



$$\underline{L} = \sum_{i=1}^{n} \underline{r}_i \times m_i(\underline{\omega} \times \underline{r}_i)$$

where $\underline{r}_i$ is p.v of m_i w.r.t O' and $\underline{\omega}$ is instantaneous angular velocity

$$\underline{L} = \sum_{i=1}^{n} m_i[r_i^2 \underline{\omega} - (\underline{r}_i \cdot \underline{\omega})\underline{r}_i] \longrightarrow (1)$$

Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along body axes x-y, z with origin at O'. Then

$$\underline{r_i} = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$$

$$\underline{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$

$$\underline{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

using these in ①

$$L_x\hat{i} + L_y\hat{j} + L_z\hat{k} = \sum_{i=1}^{n} m_i\Big[(x_i^2+y_i^2+z_i^2)(\omega_x\hat{i}+\omega_y\hat{j}+\omega_z\hat{k}) - (x_i\omega_x+y_i\omega_y+z_i\omega_z)(x_i\hat{i}+y_i\hat{j}+z_i\hat{k})\Big]$$

$$= \sum_{i=1}^{n} m_i\Big[\{(x_i^2+y_i^2+z_i^2)\omega_x - (x_i\omega_x+y_i\omega_y+z_i\omega_z)x_i\}\hat{i}$$
$$+\{(x_i^2+y_i^2+z_i^2)\omega_y - (x_i\omega_x+y_i\omega_y+z_i\omega_z)y_i\}\hat{j}$$
$$+\{(x_i^2+y_i^2+z_i^2)\omega_z - (x_i\omega_x+y_i\omega_y+z_i\omega_z)z_i\}\hat{k}\Big]$$

$$= \sum m_i\Big[\{(y_i^2+z_i^2)\omega_x - x_iy_i\omega_y - x_iz_i\omega_z\}\hat{i}$$
$$+\{(x_i^2+z_i^2)\omega_y - x_iy_i\omega_x - y_iz_i\omega_z\}\hat{j}$$
$$+\{(x_i^2+y_i^2)\omega_z - z_ix_i\omega_x - z_iy_i\omega_y\}\hat{k}\Big]$$

$$= \Big[\sum m_i(y_i^2+z_i^2)\omega_x + \Big(-\sum x_iy_im_i\Big)\omega_y + \Big(-\sum m_ix_iz_i\Big)\omega_z\Big]\hat{i}$$
$$+\Big[\sum m_i(x_i^2+z_i^2)\omega_y + \Big(-\sum y_ix_im_i\Big)\omega_x + \Big(-\sum y_iz_im_i\Big)\omega_z\Big]\hat{j}$$
$$+\Big[\sum m_i(x_i^2+y_i^2)\omega_z + \Big(-\sum z_ix_im_i\Big)\omega_x + \Big(-\sum z_iy_im_i\Big)\omega_y\Big]\hat{k}$$

$$L_x\hat{i}+L_y\hat{j}+L_z\hat{k} = (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\hat{i}$$
$$+(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\hat{k}$$
$$+(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)\hat{k}$$

Comparing co-efficients of $\hat{i}, \hat{j}, \hat{k}$

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \longrightarrow (2)$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \longrightarrow (3)$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \longrightarrow (4)$$

writing these equations in matrix form

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$[\underline{L}] = [I][\underline{\omega}] \longrightarrow (5)$$

where $[I]$ = Inertia matrix

$[\underline{\omega}]$ = Angular velocity matrix

$[\underline{L}]$ = Angular momentum matrix

Remarks # In (5) $[I]$ is an operator

which acts on a Column vector $[\underline{\omega}]$ and gives a physically new vector $[\underline{L}]$. Unlike the operator of rotation $[I]$, is not restricted to any orthogonality conditions.

Rotational K.E about C.M or Stationary Point and Relation $T_{rot} = \frac{1}{2}\underline{\omega}\cdot\underline{L} = \frac{1}{2}\underline{\omega}\cdot\{I\}\cdot\underline{\omega}$

Problem# Prove that the rotational K.E of rigid body about or w.r.t body axes at a Stationary point or at c.m (which may be fixed or translating) is given by

$$T_{rot} = \frac{1}{2}\underline{\omega}\cdot\underline{L} = \frac{1}{2}\underline{\omega}\cdot\{I\}\cdot\underline{\omega}$$

where $\{I\}$ is inertia tensor

Sol# Consider a rigid body consisting of n particle $m_1, m_2 - - - m_n$. Suppose body rotates about a point o' which may be stationary or c.m (in case of centre it may be translating). Let $\underline{r}_\alpha$ be p.v of αth particle relative to body axes $x-y-z$ at o' and $\underline{v}_\alpha$ be its relative velocity w.r.t o'. ~~Then~~ as seen in fixed system. Then

$$T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha(\underline{\omega}\times\underline{r}_\alpha)^2 \rightarrow (1)$$

where $\underline{\omega}$ is instantaneous angular velocity about o'

Also about o' angular momentum of the body is given by

$$\underline{L} = \sum_{\alpha} m_\alpha \underline{r}_\alpha \times (\underline{\omega} \times \underline{r}_\alpha) \longrightarrow ②$$

From ①

$$T_{rot} = \frac{1}{2} \sum_{\alpha} m_\alpha (\underline{\omega} \times \underline{r}_\alpha)^2$$

$$= \frac{1}{2} \sum_{\alpha} m_\alpha (\underline{\omega} \times \underline{r}_\alpha) \cdot (\underline{\omega} \times \underline{r}_\alpha)$$

$$= \frac{1}{2} \sum_{\alpha} m_\alpha \left[\omega^2 r_\alpha^2 - (\underline{\omega} \cdot \underline{r}_\alpha)^2 \right] \quad \because \left[(A \times B)^2 = A^2 B^2 - (A \cdot B)^2 \right]$$

expressing it in tensorial form by making use of components ω_i and $r_{\alpha,i}$ of $\underline{\omega}$ & $\underline{r}_\alpha$. Also $\underline{r}_\alpha = [x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}]$ in the body system. So we can write

$$r_{\alpha,i} = x_{\alpha,i}$$

$$T_{rot} = \frac{1}{2} \sum_{\alpha} m_\alpha \left[\left(\sum_{i=1}^{3} \omega_i^2 \right) \left(\sum_{k=1}^{3} x_{\alpha,k}^2 \right) - \left(\sum_{i=1}^{3} \omega_i x_{\alpha,i} \right) \left(\sum_{j=1}^{3} \omega_j x_{\alpha,j} \right) \right]$$

writing $\omega_i = \sum_{j=1}^{3} \omega_j \delta_{ij}$

$$T_{rot} = \frac{1}{2} \sum_{\alpha} m_\alpha \left[\sum_{i=1}^{3} \omega_i \sum_{j=1}^{3} \omega_j \delta_{ij} \left(\sum_{k=1}^{3} x_{\alpha,k}^2 \right) - \sum_{i=1}^{3} \sum_{j=1}^{3} \omega_i \omega_j x_{\alpha,i} x_{\alpha,j} \right]$$

$$= \frac{1}{2} \sum_{i,j} \omega_i \omega_j \sum_{\alpha} m_\alpha \left[\delta_{ij} \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right] \rightarrow ③$$

If we define the (i,j) element of sum over α to be I_{ij}. Then

$$\boxed{I_{ij} = \sum_{\alpha} m_\alpha \left[\delta_{ij} \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right]} \rightarrow ④$$

Equation ③ becomes

$$\boxed{T_{rot} = \frac{1}{2}\sum_{i,j} I_{ij}\,\omega_i\omega_j} \longrightarrow (5)$$

Similarly ith component of $\underline{L}$ i.e L_i in tensorial form from equation (2) can be written as

$$\underline{L} = \sum_{\alpha} m_\alpha\, \underline{r}_\alpha \times (\underline{\omega} \times \underline{r}_\alpha)$$

$$= \sum_{\alpha} m_\alpha \left[r_\alpha^2\, \underline{\omega} - \underline{r}_\alpha (\underline{\omega} \cdot \underline{r}_\alpha) \right]$$

$$\Rightarrow L_i = \sum_{\alpha} m_\alpha \left[r_\alpha^2\, \omega_i - x_{\alpha,i} \sum_{j=1}^{3} x_{\alpha,j}\,\omega_j \right]$$

$$= \sum_{\alpha} m_\alpha \left[\omega_i \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} \sum_{j=1}^{3} x_{\alpha,j}\,\omega_j \right]$$

putting $\omega_i = \sum_{j=1}^{3} \delta_{ij}\,\omega_j$

$$L_i = \sum_{\alpha} m_\alpha \left[\sum_{j=1}^{3} \delta_{ij}\,\omega_j \left(\sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} \sum_{j=1}^{3} x_{\alpha,j}\,\omega_j \right) \right]$$

$$= \sum_{j} \omega_j \sum_{\alpha} m_\alpha \left[\delta_{ij} \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j} \right]$$

$$\boxed{L_i = \sum_{j} I_{ij}\,\omega_j} \longrightarrow (6)$$

where I_{ij} are components of tensor $\{I\}$, which is called inertia tensor. So in dot product of dyadic $\{I\}$ (nonian form)

$$\underline{L} = \{I\} \cdot \underline{\omega} \longrightarrow (7)$$

Now multiplying (6) ~~by ω_i and summing~~ over i

$$\frac{1}{2}\sum_{i=1}^{3} \omega_i L_i = \frac{1}{2}\sum_{ij} I_{ij}\,\omega_j\,\omega_i = T_{rot} \quad \text{by } ⑤$$

$$T_{rot} = \frac{1}{2}\,\underline{\omega}\cdot\underline{L} \qquad \text{proved}$$

By ⑦ $\underline{L} = \{I\}\cdot\underline{\omega}$

$$T_{rot} = \frac{1}{2}\,\underline{\omega}\cdot\{I\}\cdot\underline{\omega}$$

General Motion of Rigid Taking origin of Body axes at c.m and Proof of $K.E = T = \frac{1}{2}\underline{V}\cdot\underline{P} + \frac{1}{2}\underline{\omega}\cdot\underline{L}_c$

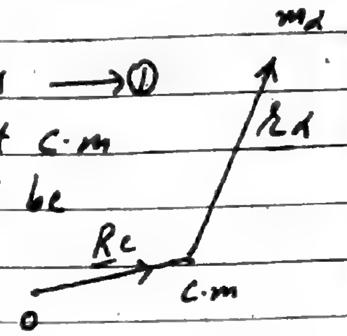
Problem# Prove that if c.m of rigid body in general motion is taken as the origin of body axes, then total K.E of rigid body is

$$T = \frac{1}{2}\underline{V}\cdot\underline{P} + \frac{1}{2}\underline{\omega}\cdot\underline{L}_c$$

where $\underline{V}$ is instantaneous velocity of translation w.r.t fixed Co-ordinate system, $\underline{P}$ is linear momentum of the body, $\underline{\omega}$ is instantaneous angular velocity of the body about c.m and $\underline{L}_c$ is angular momentum of the body about c.m.

Sol Consider a rigid body consisting of n particles of masses $m_1, m_2, \ldots m_n$. ~~Let $\underline{V}$ ... instantaneous~~ velocity of translation

w.r.t fixed Co-ordinate system. Then c.m will also translate with velocity $\underline{V}$. Suppose the body rotates with an instantaneous angular velocity $\underline{\omega}$ about c.m. The the instantaneous velocity $\underline{v}_\alpha$, of α th particle in the fixed system is given by



$$\underline{V}_\alpha = \underline{V} + \underline{\omega} \times \underline{r}_\alpha \longrightarrow (1)$$

where $\underline{r}_\alpha$ is p.v of m_α w.r.t c.m

K.E of αth particle will be

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2$$

Total K.E of the body

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left(\underline{V} + \underline{\omega} \times \underline{r}_\alpha\right)^2$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \left(\underline{V} + \underline{\omega} \times \underline{r}_\alpha\right)\cdot\left(\underline{V} + \underline{\omega} \times \underline{r}_\alpha\right)$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \left[\underline{V}\cdot\underline{V} + \underline{V}\cdot\underline{\omega}\times\underline{r}_\alpha + \underline{V}\cdot\underline{\omega}\times\underline{r}_\alpha + (\underline{\omega}\times\underline{r}_\alpha)^2\right]$$

$$= \frac{1}{2}\sum_\alpha m_\alpha V^2 + \sum_\alpha \underline{V}\cdot\underline{\omega}\times m_\alpha \underline{r}_\alpha + \frac{1}{2}\sum m_\alpha (\underline{\omega}\times\underline{r}_\alpha)^2$$

$$= \frac{1}{2}\sum_\alpha m_\alpha V^2 + \underline{V}\cdot\underline{\omega}\times\sum m_\alpha \underline{r}_\alpha + \frac{1}{2}\sum m_\alpha (\underline{\omega}\times\underline{r}_\alpha)^2$$

$$= \frac{1}{2}\sum_\alpha m_\alpha V^2 + 0 + \frac{1}{2}\sum m_\alpha (\underline{\omega}\times\underline{r}_\alpha)^2 \quad \because \sum m_\alpha \underline{r}_\alpha = 0$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \underline{V}\cdot\underline{V} + \frac{1}{2}\sum_\alpha m_\alpha (\underline{\omega}\times\underline{r}_\alpha)\cdot(\underline{\omega}\times\underline{r}_\alpha)$$

$$= \frac{1}{2} M\underline{V}\cdot\underline{V} + \frac{1}{2}\sum m_\alpha (\underline{\omega}\times\underline{r}_\alpha)\cdot(\underline{\omega}\times\underline{r}_\alpha) \longrightarrow (2)$$

$M\underline{V} = \underline{P}$ — linear momentum of body $\longrightarrow$ ③

$(\underline{\omega} \times \underline{r}_\alpha)\cdot(\underline{\omega} \times \underline{r}_\alpha) = \underline{\omega}\cdot\underline{r}_\alpha \times (\underline{\omega} \times \underline{r}_\alpha)$

$\because \underline{a} \times \underline{b} \cdot \underline{c} = \underline{a} \cdot \underline{b} \times \underline{c}$

dot and cross-product can be interchanged in scalar triple product

so $\frac{1}{2}\sum\limits_\alpha m_\alpha (\underline{\omega} \times \underline{r}_\alpha)\cdot(\underline{\omega} \times \underline{r}_\alpha)$

$= \frac{1}{2}\sum m_\alpha \left(\underline{\omega}\cdot\underline{r}_\alpha \times (\underline{\omega} \times \underline{r}_\alpha)\right)$

$= \frac{1}{2}\,\underline{\omega}\cdot\sum\limits_\alpha \underline{r}_\alpha \times m_\alpha (\underline{\omega} \times \underline{r}_\alpha)$

$= \frac{1}{2}\,\underline{\omega}\cdot\underline{L}_c \qquad \longrightarrow$ ④

putting ③ & ④ in ②

$$\boxed{T = \frac{1}{2}\,\underline{V}\cdot\underline{P} + \frac{1}{2}\,\underline{\omega}\cdot\underline{L}_c}$$ Proved

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$$[\dot{\underline{L}}] = [\underline{\omega} \times \underline{L}] + [I][\dot{\underline{\omega}}]$$

★ **Problem#** Show that in matrix notation

$$[\dot{\underline{L}}] = [\underline{\omega} \times \underline{L}] + [I][\dot{\underline{\omega}}]$$

where $[I]$ is the inertia matrix and $\underline{\omega}$, $\underline{L}$ have their usual meaning

Sol# Consider a rigid body of n particle rotating about a stationary point or in ~~[illegible]~~ , axes x-y-z are

taken at c.m.

If $\underline{r}_i$ is p.v. of ith particle relative to body system, then angular momentum is

$$\underline{L} = \sum \underline{r}_i \times m_i \underline{v}_i$$

Let ω be instantaneous angular velocity, then

$$\underline{v}_i = \underline{\omega} \times \underline{r}_i$$

$$\underline{L} = \sum_{i=1}^{n} \underline{r}_i \times m_i \underline{v}_i$$

Diff w.r.t. t

$$\frac{d\underline{L}}{dt} = \sum_{i=1}^{n} \dot{\underline{r}}_i \times m_i \underline{v}_i + \sum_{i=1}^{n} \underline{r}_i \times (m_i \dot{\underline{v}}_i)$$

$$= \sum_{i=1}^{n} \underline{v}_i \times m_i \underline{v}_i + \sum_{i=1}^{n} \underline{r}_i \times m_i \frac{d}{dt}(\underline{\omega} \times \underline{r}_i)$$

$$= \underline{0} + \sum_i \underline{r}_i \times m_i(\underline{\omega} \times \dot{\underline{r}}_i + \dot{\underline{\omega}} \times \underline{r}_i)$$

$$\dot{\underline{L}} = \sum_i \underline{r}_i \times m_i(\underline{\omega} \times \dot{\underline{r}}_i) + \sum \underline{r}_i \times m_i(\dot{\underline{\omega}} \times \underline{r}_i)$$

$$= \sum_i \underline{r}_i \times m_i(\underline{\omega} \times \underline{v}_i) + \sum m_i(r_i^2 \dot{\underline{\omega}} - (\underline{r}_i \cdot \dot{\underline{\omega}})\underline{r}_i)$$

$$= \sum_i \underline{r}_i \times m_i\{\underline{\omega} \times (\underline{\omega} \times \underline{r}_i)\} + \sum m_i\{r_i^2 \dot{\underline{\omega}} - (\underline{r}_i \cdot \dot{\underline{\omega}})\underline{r}_i\}$$

$$= \sum_i m_i \underline{r}_i \times \{(\underline{\omega} \cdot \underline{r}_i)\underline{\omega} - (\underline{\omega} \cdot \underline{\omega})\underline{r}_i\} + \sum m_i[r_i^2 \dot{\underline{\omega}} - (\underline{r}_i \cdot \dot{\underline{\omega}})\underline{r}_i]$$

$$= \sum_i m_i \underline{r}_i \times (\underline{\omega} \cdot \underline{r}_i)\underline{\omega} - 0 + \sum m_i[r_i^2 \dot{\underline{\omega}} - (\underline{r}_i \cdot \dot{\underline{\omega}})\underline{r}_i]$$

$$= -\sum_i m_i \underline{\omega} \times (\underline{\omega} \cdot \underline{r}_i)\underline{r}_i + \sum_i m_i[r_i^2 \dot{\underline{\omega}} - (\underline{r}_i \cdot \dot{\underline{\omega}})\underline{r}_i]$$

$\dot{\omega}$(... 3 −) + ...(... 3) +

$$= \sum_i m_i \left[-\underline{\omega} \times (\underline{\omega} \cdot \underline{r}_i)\underline{r}_i \right] + \sum \left[r_i^2 \dot{\underline{\omega}} - (\underline{r}_i \cdot \dot{\underline{\omega}})\underline{r}_i \right] m_i$$

$$= \sum_i m_i \left[\underline{\omega} \overset{0}{\times} (\underline{r}_i \cdot \underline{r}_i)\underline{\omega} - \underline{\omega} \times (\underline{\omega} \cdot \underline{r}_i)\underline{r}_i \right]$$
$$+ \sum m_i \left[r_i^2 \dot{\underline{\omega}} - (\underline{r}_i \cdot \dot{\omega})\underline{r}_i \right]$$

$$= \sum_i m_i \; \underline{\omega} \times \left[(\underline{r}_i \cdot \underline{r}_i)\underline{\omega} - (\underline{\omega} \cdot \underline{r}_i)\underline{r}_i \right]$$
$$+ \sum_i m_i \left[r_i^2 \dot{\underline{\omega}} - (\underline{r}_i \cdot \dot{\omega})\underline{r}_i \right]$$

$$= \sum_i m_i \, \underline{\omega} \times \left[\underline{r}_i \times (\underline{\omega} \times \underline{r}_i) \right] + \sum m_i \left[r_i^2 \dot{\underline{\omega}} - (\underline{r}_i \cdot \dot{\omega})\underline{r}_i \right]$$

$$= \sum_i \underline{\omega} \times m_i (\underline{r}_i \times \underline{v}_i) + \sum_i m_i \left[r_i^2 \dot{\underline{\omega}} - (r_i \cdot \dot{\omega}) r_i \right]$$

$$= \underline{\omega} \times \underline{L} + \sum m_i \left[r_i^2 \dot{\underline{\omega}} - (r_i \cdot \dot{\underline{\omega}}) \underline{r}_i \right].$$

Now
$$\underline{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$$
$$\underline{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$
$$\underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\frac{\underline{L}}{\delta t} = \underline{\omega} \times \underline{L} + \sum_i m_i \Big\{ (x_i^2 + y_i^2 + z_i^2)(\dot{\omega}_x \hat{i} + \dot{\omega}_y \hat{j} + \dot{\omega}_z \hat{k})$$
$$- (x_i \dot{\omega}_x + y_i \dot{\omega}_y + z_i \dot{\omega}_z)(x_i \hat{i} + y_i \hat{j} + z_i \hat{k}) \Big\}$$

$$= \underline{\omega} \times \underline{L} + \Big\{ \sum m_i (y_i^2 + z_i^2)\dot{\omega}_x + \Big(-\sum_i m_i x_i y_i\Big)\dot{\omega}_y + \Big(-\sum m_i x_i z_i\Big)\dot{\omega}_z \Big\} \hat{i}$$

$$+ \Big\{ \sum m_i (x_i^2 + z_i^2)\dot{\omega}_y + \Big(-\sum_i m_i y_i x_i\Big)\dot{\omega}_x + \Big(-\sum m_i y_i z_i\Big)\dot{\omega}_z \Big\} \hat{j}$$

$$+ \Big\{ \Big(-\sum m_i z_i x_i\Big)\dot{\omega}_x + \Big(-\sum m_i z_i y_i\Big)\dot{\omega}_y + \sum m_i (x_i^2 + y_i^2)\dot{\omega}_z \Big\} \hat{k}$$

let $P = \underline{\omega} \times \underline{L}$

$$\dot{\underline{L}} = \underline{\omega} \times \underline{L} + (I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z)\hat{i} + \hat{j}(I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z)$$
$$+ (I_{zx}\dot{\omega}_x + I_{zx}\dot{\omega}_y + I_{zz}\dot{\omega}_z)\hat{k}$$

$$\dot{L}_x\hat{i} + \dot{L}_y\hat{j} + \dot{L}_z = \overset{P_x}{(\underline{\omega}\times\underline{L})_x}\hat{i} + \overset{P_y}{(\underline{\omega}\times\underline{L})_y}\hat{j} + \overset{P_z}{(\underline{\omega}\times\underline{L})_z}\hat{k}$$
$$+ (I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z)\hat{i} + \hat{j}(I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z)$$
$$+ (I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z)$$

Comparing both sides

$$\dot{L}_x = \overset{P_x}{(\underline{\omega}\times\underline{L})_x} + I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \longrightarrow ①$$

$$\dot{L}_y = P_y + I_{yx}\omega_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \longrightarrow ②$$

$$\dot{L}_z = P_z + I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z \longrightarrow ③$$

writing ①, ② & ③ in matrix form

$$\begin{bmatrix} \dot{L}_x \\ \dot{L}_y \\ \dot{L}_z \end{bmatrix} = \begin{bmatrix} P_x + I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ P_y + I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ P_z + I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z \end{bmatrix}$$

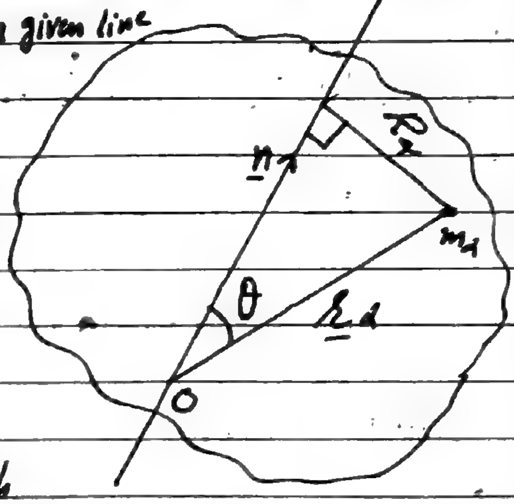
$$= \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} + \begin{bmatrix} I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z \\ I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z \end{bmatrix}$$

$$[\dot{\underline{L}}] = [\underline{P}] + \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = [\underline{\omega}\times\underline{L}] + [\underline{I}][\dot{\underline{\omega}}]$$

proved.

Moment of Inertia of a Rigid Body About a line in Vector form

Problem # Derive expression for moment of inertia of a rigid body about a line and write it in tensorial Form. Also find the moment of inertia about line in terms of moment of inerti about axes, product of inertia and direction cosines of given line. OR *Find moment of inertia of a rigid body about a given line



Sol Consider a rigid body consisting of n particles and rotating about a stationary point O with instaneous angular velocity ω. Let $\underline{n}$ be unit vector along a line about which moment of inertia is required. Let r_α be p.v of αth particle relative to O and R_α be its $\perp$ar distance from line. Then

$$R_\alpha = r_\alpha \sin\theta$$

$$= |\underline{n}|\, r_\alpha \sin\theta \qquad \because |\underline{n}| = 1$$

$$= |\underline{r}_\alpha \times \underline{n}|$$

$$R_\alpha^2 = (\underline{r}_\alpha \times \underline{n})^2$$

(i) Now moment of inertia for α th particle about L is $m_\alpha R_\alpha^2 = m_\alpha (\underline{r}_\alpha \times \underline{n})^2$

The moment of inertia of the whole body is

$$I = \sum_{\alpha} m_\alpha (\underline{r}_\alpha \times \underline{n})^2$$

$$= \sum_{\alpha} m_\alpha (\underline{r}_\alpha \times \underline{n}) \cdot (\underline{r}_\alpha \times \underline{n}) \longrightarrow (1)$$

which is moment of inertia about L in vector form

We not that moment of inertia depends on the origin O (which determines the p.v $\underline{r}_\alpha$) and on $\underline{n}$ which gives the orientation or direction of the axis. To stress the dependence of I on O and $\underline{n}$ we sometimes write I as $I(O, \underline{n})$

Tensorial Form

If n_i. $i=1,2,3$ are Components of unitvector $\underline{n}$ (i.e D. Cosines of $\underline{n}$), then

$$(\underline{n} \times \underline{r}_\alpha)_k = \epsilon_{ijk}\, n_i\, r_{\alpha,j}$$

where $r_{\alpha,j} = x_{\alpha,j}$ $(j=1,2,3)$ are Components of $\underline{r}_\alpha$

Remembering that dot product

$$\underline{A} \cdot \underline{A} = A_k A_k \quad (k \text{ dummy})$$
$$= \sum_{k=1}^{3} A_k A_k$$

we can write (1) as

$$I = \sum_{\alpha} m_\alpha (\underline{r}_\alpha \times \underline{n})_k (\underline{r}_\alpha \times \underline{n})_k \quad k \text{ dummy i.e Summation index}$$

$$I = \sum_\alpha m_\alpha \, \epsilon_{ijk} n_i x_{\alpha,j} \, \epsilon_{lmk} n_l x_{\alpha,m}$$

$$= \sum_\alpha m_\alpha \, n_i n_l \, \epsilon_{ijk} \epsilon_{lmk} \, x_{\alpha,j} x_{\alpha,m}$$

on R.H.S i, j, k, l, m are dummy indices.
using the relation

$$\epsilon_{ijk} \epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$I = \sum_\alpha m_\alpha (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) n_i n_l x_{\alpha,j} x_{\alpha,m}$$

$$= \sum_\alpha m_\alpha (\delta_{il}\delta_{jm} n_i n_l x_{\alpha,j} x_{\alpha,m}) - \sum_\alpha m_\alpha (\delta_{im}\delta_{jl} n_i n_l x_{\alpha,j} x_{\alpha,m})$$

$$= \sum_\alpha m_\alpha n_i n_i x_{\alpha,k} x_{\alpha,k} - \sum_\alpha m_\alpha n_i n_j x_{\alpha,i} x_{\alpha,j}$$

(double dummy can not be used in an expression, so k is used)

$$= \sum_\alpha m_\alpha (n_i n_i x^2_{\alpha,k} - n_i n_j x_{\alpha,i} x_{\alpha,j})$$

using $n_i = \sum n_j \delta_{ij} = n_j \delta_{ij}$ $\quad x^2_{\alpha,k} = \sum x_{\alpha,k} x_{\alpha,k} = \lambda^2_\alpha$

$$I = \sum_\alpha m_\alpha [n_i n_j \delta_{ij} \lambda^2_\alpha - n_i n_j x_{\alpha,i} x_{\alpha,j}]$$

$$\sum_\alpha m_\alpha n_i n_j [\lambda^2_\alpha \delta_{ij} - x_{\alpha,i} x_{\alpha,j}]$$

$$= \sum_\alpha n_i n_j \, m_\alpha [\lambda^2_\alpha \delta_{ij} - x_{\alpha,i} x_{\alpha,j}] \longrightarrow (2)$$

$$\boxed{I = \sum_\alpha n_i n_j I_{ij}} \longrightarrow ③$$

where

$$\boxed{I_{ij} = \sum_\alpha m_\alpha \left[r_\alpha^2 \delta_{ij} - x_{\alpha,i}\, x_{\alpha,j} \right]} \rightarrow ④$$

where I_{ij} is called inertia tensor. Its components can be calculated as

$$I_{xx} = I_{11} = \sum_\alpha m_\alpha \left[r_\alpha^2 \delta_{11} - x_{\alpha,1}^2 \right]$$

$$= \sum_\alpha m_\alpha \left[r_{\alpha,k}^2 - x_{\alpha,1}^2 \right]$$

$$= \sum_\alpha m_\alpha \left[x_{\alpha,1}^2 + x_{\alpha,2}^2 + x_{\alpha,3}^2 - x_{\alpha,1}^2 \right]$$

$$= \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2 \right) = \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2 \right)$$

Similarly

$$I_{yy} = I_{22} = \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2 \right) = \sum_\alpha m_\alpha \left(x_\alpha^2 + z_\alpha^2 \right)$$

$$I_{zz} = I_{33} = \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2 \right) = \sum_\alpha m_\alpha \left(x_\alpha^2 + y_\alpha^2 \right)$$

$$I_{xy} = I_{12} = \sum_\alpha m_\alpha \left[r_{\alpha,k}^2 \delta_{12} - x_{\alpha,1}\, x_{\alpha,2} \right]$$

$$= \sum_\alpha m_\alpha \left[0 - x_\alpha y_\alpha \right]$$

$$= -\sum_\alpha m_\alpha x_\alpha y_\alpha$$

Similarly

$$I_{xz} = I_{13} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$$

$$I_{yz} = I_{23} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$$

Moment of Inertia and Inertia Tensor in Dyadic Form *

From ①

$$I = \sum_\alpha m_\alpha (\underline{r}_\alpha \times \underline{n}) \cdot (\underline{r}_\alpha \times \underline{n})$$

$$= \sum_\alpha m_\alpha \left[r_\alpha^2 \, \underline{n} \cdot \underline{n} - (\underline{r}_\alpha \cdot \underline{n})(\underline{r}_\alpha \cdot \underline{n}) \right]$$

$$= \sum_\alpha m_\alpha \left[r_\alpha^2 \, \underline{n} \cdot \underline{n} - (\underline{n} \cdot \underline{r}_\alpha)(\underline{n} \cdot \underline{r}_\alpha) \right]$$

Now by double dot product of two dyadic $\underline{r}_\alpha \underline{r}_\alpha$ $\underline{n}\,\underline{n}$, we have

$$\underline{r}_\alpha \underline{r}_\alpha : \underline{n}\,\underline{n} = (\underline{n} \cdot \underline{r}_\alpha)(\underline{r}_\alpha \cdot \underline{n})$$

(2)

$$= \underline{n} \cdot \underline{r}_\alpha \underline{r}_\alpha \cdot \underline{n}$$

(3) Also for unit dyad $\underline{I} = \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}$, we have

$$\underline{I} \cdot \underline{n} = \underline{n} \cdot \underline{I} = \underline{n}$$

So using these

$$I = \sum_\alpha m_\alpha \left[r_\alpha^2 \, \underline{n} \cdot \underline{I} \cdot \underline{n} - \underline{n} \cdot \underline{r}_\alpha \underline{r}_\alpha \cdot \underline{n} \right]$$

$$= \underline{n} \cdot \left[\sum_\alpha m_\alpha \left(r_\alpha^2 \underline{I} - \underline{r}_\alpha \underline{r}_\alpha \right) \right] \cdot \underline{n}$$

$$\boxed{I = \underline{n} \cdot \underline{I} \cdot \underline{n}} \longrightarrow ⑤$$

where $\underline{I} = \sum m_\alpha \left[r_\alpha^2 \underline{I} - \underline{r}_\alpha \underline{r}_\alpha \right]$

is Inertia tensor in dyadic form. In nonion form it can be written as

$$\underline{I} = \sum_\alpha m_\alpha \left[r_\alpha^2 \underline{1} - r_\alpha r_\alpha \right]$$

$$= \sum_\alpha m_\alpha \Big[(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)(\hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}) - (x_\alpha \hat{i} + y_\alpha \hat{j} + z_\alpha \hat{k})(x_\alpha \hat{i} + y_\alpha \hat{j} + z_\alpha \hat{k}) \Big]$$

$$= \sum_\alpha m_\alpha \Big[(x_\alpha^2 + y_\alpha^2 + z_\alpha^2)(\hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}) - (x_\alpha^2 \hat{i}\hat{i} + x_\alpha y_\alpha \hat{i}\hat{j} + x_\alpha z_\alpha \hat{i}\hat{k} + x_\alpha y_\alpha \hat{j}\hat{i} + y_\alpha^2 \hat{j}\hat{j} + y_\alpha z_\alpha \hat{j}\hat{k} + z_\alpha x_\alpha \hat{k}\hat{i} + z_\alpha y_\alpha \hat{k}\hat{j} + z_\alpha^2 \hat{k}\hat{k}) \Big]$$

$$= \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)\hat{i}\hat{i} + \left(-\sum m_\alpha x_\alpha y_\alpha\right)\hat{i}\hat{j} + \left(-\sum m_\alpha x_\alpha z_\alpha\right)\hat{i}\hat{k}$$

$$+ \left(-\sum m_\alpha y_\alpha x_\alpha\right)\hat{j}\hat{i} + \sum_\alpha m_\alpha (x_\alpha^2 + z_\alpha^2)\hat{j}\hat{j} + \left(-\sum m_\alpha y_\alpha z_\alpha\right)\hat{j}\hat{k}$$

$$+ \left(-\sum m_\alpha z_\alpha x_\alpha\right)\hat{k}\hat{i} + \left(-\sum m_\alpha z_\alpha y_\alpha\right)\hat{k}\hat{j} + \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)$$

$$= I_{xx}\hat{i}\hat{i} + I_{xy}\hat{i}\hat{j} + I_{xz}\hat{i}\hat{k}$$

$$+ I_{yx}\hat{j}\hat{i} + I_{yy}\hat{j}\hat{j} + I_{yz}\hat{j}\hat{k}$$

$$+ I_{zx}\hat{k}\hat{i} + I_{zy}\hat{k}\hat{j} + I_{zz}\hat{k}\hat{k}$$

$$= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \quad \text{nonion form}$$

These Components of dyadic $\underline{I}$ are denoted by I_{ij}

, where $x = x_1$ $y = y_1$ $z = z_1$ are taken as Co-ordinate axes.

Now dot product of dyadic $\underline{I}$ with vector $\underline{C}$ is defined as

$$\underline{I} \cdot \underline{C} = \sum_j I_{ij} C_j = I_{ij} C_j$$

$$\& \quad \underline{C} \cdot \underline{I} = \sum_{j=1}^{3} C_j I_{ji} = C_j I_{ji}$$

and scalar S formed by $\underline{A} \cdot \underline{I} \cdot \underline{B}$ where $\underline{A}, \underline{B}$ are vectors and $\underline{I}$ is dyadic or dyadic Tensor, is given by

$$S = \underline{A} \cdot \underline{I} \cdot \underline{B}$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i I_{ij} B_j$$

Using this we have

$$\overset{n}{I} = \underline{n} \cdot \underline{I} \cdot \underline{n} = \sum_{i=1}^{3} \sum_{j=1}^{3} n_i I_{ij} n_j$$

$$= \sum_{i=1}^{3} \left[n_i I_{i1} n_1 + n_i I_{i2} n_2 + n_i I_{i3} n_3 \right]$$

$$= \sum_{i=1}^{3} n_i I_{i1} n_1 + \sum_{i=1}^{3} n_i I_{i2} n_2 + \sum_{i=1}^{3} n_i I_{i3} n_3$$

$$= (n_1 I_{11} n_1 + n_2 I_{21} n_1 + n_3 I_{31} n_1) + (n_1 I_{12} n_2 + n_2 I_{22} n_2 + n_3 I_{32} n_2)$$

$$+ (n_1 I_{13} n_3 + n_2 I_{23} n_3 + n_3 I_{33} n_3)$$

$$= I_{11} n_1^2 + I_{22} n_2^2 + I_{33} n_3^2 + 2 I_{12} n_1 n_2 + 2 I_{13} n_1 n_3 + 2 I_{23} n_2 n_3$$

where $I_{12} = I_{21}$ etc.

[illegible]

which matches with the expansion of ③

Thus we have $I = \sum n_i n_j I_{ij} = \underline{n} \cdot \underline{I} \cdot \underline{n}$

where $\underline{n}$ is unit vector along the axis about which moment of inertia is calculated.

Note # We note that from

$$\underline{I} = I_{xx}\hat{i}\hat{i} + I_{xy}\hat{i}\hat{j} + I_{xz}\hat{i}\hat{k}$$
$$+ I_{yx}\hat{j}\hat{i} + I_{yy}\hat{j}\hat{j} + I_{yz}\hat{j}\hat{k}$$
$$+ I_{zx}\hat{k}\hat{i} + I_{zy}\hat{k}\hat{j} + I_{zz}\hat{k}\hat{k}$$

$\hat{i}\cdot\underline{I}\cdot\hat{i} = I_{xx}$ $\quad \hat{j}\cdot\underline{I}\cdot\hat{j} = I_{yy}$ $\quad \hat{k}\cdot\underline{I}\cdot\hat{k} = I_{zz}$

$\hat{i}\cdot\underline{I}\cdot\hat{j} = I_{xy}$ $\quad \hat{i}\cdot\underline{I}\cdot\hat{k} = I_{xz}$

$\hat{j}\cdot\underline{I}\cdot\hat{i} = I_{yx}$ $\quad \hat{k}\cdot\underline{I}\cdot\hat{i} = I_{zx}$

$\hat{j}\cdot\underline{I}\cdot\hat{k} = I_{yz}$ $\quad \hat{k}\cdot\underline{I}\cdot\hat{j} = I_{zy}$

So $\underline{I}$ can be written as

$$\underline{I} = (\hat{i}\cdot\underline{I}\cdot\hat{i})\hat{i}\hat{i} + (\hat{i}\cdot\underline{I}\cdot\hat{j})\hat{i}\hat{j} + (\hat{i}\cdot\underline{I}\cdot\hat{k})\hat{i}\hat{k}$$
$$+ (\hat{j}\cdot\underline{I}\cdot\hat{i})\hat{j}\hat{i} + (\hat{j}\cdot\underline{I}\cdot\hat{j})\hat{j}\hat{j} + (\hat{j}\cdot\underline{I}\cdot\hat{k})\hat{j}\hat{k}$$
$$+ (\hat{k}\cdot\underline{I}\cdot\hat{i})\hat{k}\hat{i} + (\hat{k}\cdot\underline{I}\cdot\hat{j})\hat{k}\hat{j} + (\hat{k}\cdot\underline{I}\cdot\hat{k})\hat{k}\hat{k}$$

Moments of Inertia about Line In Terms of momenta about axes and Product of Inertia

from equation ② [illegible]

$$I = \sum_{i,j} n_i n_j I_{ij} = n_i n_j I_{ij}$$

$$= n_1 n_j I_{1j} + n_2 n_j I_{2j} + n_3 n_j I_{3j}$$

$$= (n_1 n_1 I_{11} + n_1 n_2 I_{12} + n_1 n_3 I_{13}) + n_2 n_1 I_{21} + n_2 n_2 I_{22}$$
$$+ n_2 n_3 I_{23} + n_3 n_1 I_{31} + n_3 n_2 I_{32} + n_3 n_3 I_{33}$$

$$= n_1^2 I_{11} + n_2^2 I_{22} + n_3^2 I_{33} + 2 n_1 n_2 I_{12} + 2 n_1 n_3 I_{13}$$
$$+ 2 n_2 n_3 I_{23}$$

which is required relation

Deduction# from

$$I = \sum_\alpha m_\alpha (\underline{r}_\alpha \times \underline{n}) \cdot (\underline{r}_\alpha \times \underline{n})$$

we can deduce the expression for K.E as

Multiplying and dividing by ω^2

$$I = \frac{\omega^2}{\omega^2} \sum_\alpha m_\alpha (\underline{r}_\alpha \times \underline{n}) \cdot (\underline{r}_\alpha \times \underline{n})$$

$$= \frac{\omega \omega}{\omega^2} \sum_\alpha m_\alpha (\underline{r}_\alpha \times \underline{n}) \cdot (\underline{r}_\alpha \times \underline{n})$$

$$= \frac{1}{\omega^2} \sum_\alpha m_\alpha (\underline{r}_\alpha \times \omega \underline{n}) \cdot (\underline{r}_\alpha \times \omega \underline{n})$$

$$= \frac{1}{\omega^2} \sum_\alpha m_\alpha (\underline{r}_\alpha \times \underline{\omega}) \cdot (\underline{r}_\alpha \times \underline{\omega}) \quad \because \omega \underline{n} = \underline{\omega}$$

But $v_\alpha = \underline{r}_\alpha \times \underline{\omega}$

$$= \frac{1}{\omega^2} \sum m_\alpha v_\alpha^2$$

$$I = \frac{2T}{\omega^2}$$

$$\Rightarrow \boxed{T = \frac{1}{2}\omega^2} \text{ deduced Result.}$$

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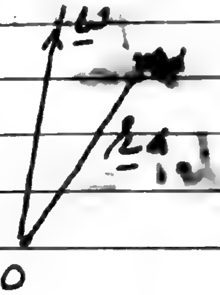
K.E & Angular Momentum and Inertia Tensor in Tensorial & Dyadic Forms

Problem # Derive expressions for K.E & angular momentum of a rigid body in tensorial and dyadic forms. Also prove that

(a) $\underline{L} = \underline{I} \cdot \underline{\omega}$ (b) $T = \frac{1}{2}\underline{\omega}\cdot\underline{L} = \frac{1}{2}\underline{\omega}\cdot\underline{I}\cdot\underline{\omega}$

Sol # suppose a rigid of n particles rotates about a stationary point O. If $\underline{r}_\alpha$ is p.v of αth particle relative to O, then angular momentum is



$$\underline{L} = \sum_\alpha \underline{r}_\alpha \times m_\alpha \underline{v}_\alpha$$

$$= \sum_\alpha m_\alpha [\underline{r}_\alpha \times (\underline{\omega} \times \underline{r}_\alpha)] \rightarrow ①$$

where $\underline{\omega}$ is instanteous angular velocity.

K E of the body

$$T = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2$$

$$= \frac{1}{2}\sum_\alpha m_\alpha [illegible] \rightarrow ②$$

From ①

$$\underline{L} = \sum m_\alpha [r_\alpha^2 \underline{\omega} - (\underline{\omega}\cdot\underline{r}_\alpha)\underline{r}_\alpha]$$

Let $r_{\alpha,i} = x_{\alpha,i}$, ω_i, L_i be components of $\underline{r}_\alpha$, $\underline{\omega}$ & $\underline{L}$. Then

$$r_\alpha^2 = \sum_{i=1}^{3} r_{\alpha,i}\, r_{\alpha,i} = \sum_{k=1}^{3} x_{\alpha,k}^2$$

$$\underline{\omega}\cdot\underline{r}_\alpha = \sum_{j=1}^{3} \omega_j r_{\alpha,j} = \omega_j x_{\alpha,j}$$

ith Component of $\underline{L}$ is

$$L_i = \sum_\alpha m_\alpha \left[\omega_i \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i}\sum_{j=1}^{3} x_{\alpha,j}\,\omega_j\right]$$

Here i is not summation index

putting $\omega_i = \sum_j \delta_{ij}\omega_j = \delta_{ij}\omega_j$

$$L_i = \sum_\alpha m_\alpha \left[\sum_{j=1}^{3}\delta_{ij}\omega_j \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i}\sum_j x_{\alpha,j}\,\omega_j\right]$$

$$= \sum_j \omega_j \sum_\alpha m_\alpha \left[\delta_{ij}\sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j}\right]$$

$$\boxed{L_i = \sum_j \omega_j I_{ij}} \Longrightarrow ③ \text{ Required Tensor form}$$

where $I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij}\sum x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j}\right]$

Dyadic form of angular Momentum

$$\underline{L} = \sum m_\alpha [r_\alpha^2 \underline{\omega} - \underline{r}_\alpha(\underline{r}_\alpha\cdot\underline{\omega})]$$

Using unit dyadic $\underline{I} = \hat{i}\hat{i}+\hat{j}\hat{j}+\hat{k}\hat{k}$

, we have

$$\underline{1}\cdot\underline{\omega} = \underline{\omega}$$

Also $\underline{r}_\alpha(\underline{r}_\alpha\cdot\underline{\omega}) = \underline{r}_\alpha\underline{r}_\alpha\cdot\underline{\omega}$ where $\underline{r}_\alpha\underline{r}_\alpha$ is dyadic

$$\underline{L} = \sum_\alpha m_\alpha\left[r_\alpha^2\,\underline{1}\cdot\underline{\omega} - \underline{r}_\alpha\underline{r}_\alpha\cdot\underline{\omega}\right]$$

$$\boxed{\underline{L} = \sum_\alpha m_\alpha\left[r_\alpha^2\,\underline{1} - \underline{r}_\alpha\underline{r}_\alpha\right]\cdot\underline{\omega}} \longrightarrow (4)$$

which is required expression in dyadic form. It can be further written as

$$\boxed{\underline{L} = \underline{I}\cdot\underline{\omega}} \longrightarrow (5)$$

where $\underline{I} = \sum_\alpha m_\alpha\left[r_\alpha^2\,\underline{1} - \underline{r}_\alpha\underline{r}_\alpha\right]$

inertia tensor in dyadic form. In matrix form (nonion) it can be written as

$$\underline{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

So

$$\underline{L} = \underline{I}\cdot\underline{\omega} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}\cdot(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})$$

$$= (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\hat{i} + (I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\hat{j}$$

$$+ (I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)\hat{k} \longrightarrow (6)$$

K·E in Tensorial and Dyadic Form

From ②

$$T = \frac{1}{2}\sum_{\alpha} m_\alpha (\underline{\omega} \times \underline{r}_\alpha)^2$$

$$= \frac{1}{2}\sum_{\alpha} m_\alpha (\underline{\omega} \times \underline{r}_\alpha)\cdot(\underline{\omega} \times \underline{r}_\alpha)$$

$$= \frac{1}{2}\sum_{\alpha} m_\alpha \left[(\underline{r}_\alpha \times \underline{\omega})\cdot(\underline{r}_\alpha \times \underline{\omega})\right]$$

$$= \frac{1}{2}\sum_{\alpha} m_\alpha \left[r_\alpha^2 \omega^2 - (\underline{\omega}\cdot\underline{r}_\alpha)^2\right]$$

$$= \frac{1}{2}\sum_{\alpha} m_\alpha \left[r_\alpha^2\, \underline{\omega}\cdot\underline{\omega} - (\underline{\omega}\cdot\underline{r}_\alpha)(\underline{\omega}\cdot\underline{r}_\alpha)\right]$$

$$\underline{\omega}\cdot\underline{\omega} = \sum \omega_i \omega_i = \omega_i^2$$

$$\underline{\omega}\cdot\underline{r}_\alpha = \sum_{i=1}^{3} \omega_i r_{\alpha,i} = \sum_{i=1}^{3} \omega_i x_{\alpha,i}$$

$$r_\alpha^2 = \sum_{k=1}^{3} r_{\alpha,k} r_{\alpha,k} = \sum_{k=1}^{3} r_{\alpha,k}^2 = \sum_{k=1}^{3} x_{\alpha,k}^2$$

$$T = \frac{1}{2}\sum_{\alpha} m_\alpha \left[\left(\sum_i \omega_i \omega_i\right)\left(\sum_{k=1}^{3} x_{\alpha,k}^2\right) - \left(\sum_i \omega_i x_{\alpha,i}\right)\left(\sum_j \omega_j x_{\alpha,j}\right)\right]$$

using $\omega_i = \sum_{j=1}^{3} \delta_{ij}\omega_j$

$$T = \frac{1}{2}\sum_{\alpha} m_\alpha \left[\sum_i \omega_i \sum_j \delta_{ij}\omega_j \left(\sum_{k=1}^{3} x_{\alpha,k}^2\right) - \left(\sum_i \omega_i x_{\alpha,i}\right)\left(\sum_j \omega_j x_{\alpha,j}\right)\right]$$

$$T = \frac{1}{2}\sum_{ij} \omega_i \omega_j \sum_{\alpha} m_\alpha \left[\delta_{ij}\left(\sum_k x_{\alpha,k}^2\right) - x_{\alpha,i}\, x_{\alpha,j}\right] \rightarrow ⑦$$

$$\boxed{T = \frac{1}{2}\sum_{i,j} I_{ij}\,\omega_i\omega_j} \longrightarrow (8)$$

where $I_{ij} = \sum_{\alpha} m_\alpha \left[\delta_{ij}\sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i}\,x_{\alpha,j}\right]$

is inertia tensor. (8) is required expression in tensorial form.

Dyadic Form

$$T = \left[\frac{1}{2}\sum m_\alpha\left[r_\alpha^2\,\omega^2 - (\underline{\omega}\cdot \underline{r}_\alpha)(\underline{r}_\alpha\cdot\underline{\omega})\right]\right]$$

$$= \frac{1}{2}\sum_\alpha m_\alpha\left[r_\alpha^2\,\underline{\omega}\cdot\underline{\omega} - \underline{\omega}\cdot\underline{r}_\alpha\underline{r}_\alpha\cdot\underline{\omega}\right]$$

$$= \frac{1}{2}\underline{\omega}\cdot\sum_\alpha m_\alpha\left[r_\alpha^2\,\underline{\omega} - \underline{r}_\alpha\underline{r}_\alpha\cdot\underline{\omega}\right]$$

$$= \frac{1}{2}\underline{\omega}\cdot\sum m_\alpha\left[r_\alpha^2\,\underline{\underline{1}}\cdot\underline{\omega} - \underline{r}_\alpha\underline{r}_\alpha\cdot\underline{\omega}\right]$$

$$= \frac{1}{2}\underline{\omega}\cdot\sum_\alpha m_\alpha\left[r_\alpha^2\,\underline{\underline{1}} - \underline{r}_\alpha\underline{r}_\alpha\right]\cdot\underline{\omega}$$

$$\boxed{T = \frac{1}{2}\,\underline{\omega}\cdot\underline{\underline{I}}\cdot\underline{\omega}} \longrightarrow (9)$$

where $\underline{\underline{I}} = \sum_\alpha m_\alpha\left[r_\alpha^2\,\underline{\underline{1}} - \underline{r}_\alpha\underline{r}_\alpha\right]$

is dyadic.

since $\underline{L} = \underline{\underline{I}}\cdot\underline{\omega}$

Therefor. $\boxed{T = \frac{1}{2}\,\underline{\omega}\cdot\underline{L}} \longrightarrow (10)$

This result can also be [illegible] directly

as $T = \frac{1}{2}\sum m_\alpha v_\alpha^2$ [illegible]

$$T = \frac{1}{2}\sum_\alpha m_\alpha \underline{v}_\alpha \cdot \underline{v}_\alpha$$

$$= \frac{1}{2}\sum_\alpha m_\alpha [\underline{v}_\alpha \cdot (\underline{\omega} \times \underline{r}_\alpha)]$$

By permuting the vectors in scalar triple product, we have

$$\underline{v}_\alpha \cdot (\underline{\omega} \times \underline{r}_\alpha)$$

$$= \underline{\omega} \cdot (\underline{r}_\alpha \times \underline{v}_\alpha)$$

so

$$T = \frac{1}{2}\sum_\alpha m_\alpha [\cdots]$$

$$= \frac{1}{2}\,\omega \cdot \sum_\alpha m_\alpha (\underline{r}_\alpha \times \underline{v}_\alpha)$$

$$T = \frac{1}{2}\underline{\omega} \cdot \underline{L}$$

...ies # 1) # $T = \frac{1}{2}\,\underline{\omega}\cdot\underline{L} = \frac{1}{2}(\underline{\omega}\cdot\underline{I}\cdot\underline{\omega})$

... be unit vector in the direction of ω so that $\underline{\omega} = \omega \underline{n}$

$$T = \frac{\omega\underline{n}\cdot\underline{I}\cdot\underline{n}\omega}{2} = \frac{\omega^2(\underline{n}\cdot\underline{I}\cdot\underline{n})}{2}$$

$$= \frac{1}{2}I\omega^2$$

where I is a scalar defined by

$$I = \underline{n}\cdot\underline{I}\cdot\underline{n} = \sum m_\alpha \left[r_\alpha^2 - (\underline{r}_\alpha \cdot \underline{n})^2\right]$$

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \cdot (\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})$$

... the ... ... are principal axes,

$I_{xy} = I_{yx} = I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$ and

$$\underline{L} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k})$$

$$= I_{xx}\omega_x \hat{i} + I_{yy}\omega_y \hat{j} + I_{zz}\omega_z \hat{k}$$

If the system is rotating about a fixed pricipal axes say z- axis, then

$$\underline{L} = I_{zz}\omega_z \hat{k} \qquad \because \omega_y = \omega_x = 0$$
$$I_{yy} = I_{xx} = 0$$

$$= I_{zz}\,\underline{\omega}$$

$\Rightarrow$ In this angular velocity vector and angular momentum are parallel

$$L = |\underline{L}| = I_{zz}\omega = Mk^2\dot{\theta}$$

where k is the radius of gyration of the system about the fixed axis.

3)# $T = \frac{1}{2}(\underline{\omega}\cdot\underline{I}\cdot\underline{\omega})$

$$= \frac{1}{2}[I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y$$
$$+ 2I_{yz}\omega_y\omega_z + 2I_{xz}\omega_x\omega_z]$$

If the co-ordinate axes are principal axes, then

$$T = \frac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2)$$

If the system is rotating about a fixed axis say z - axis, then

$$T = \frac{1}{2}I_{zz}\omega_z^2 \qquad \frac{1}{2}I\dot{\theta}^2$$

$$= \frac{1}{2}Mk^2\dot{\theta}^2$$

Diagonalisation of 3×3 Matrices

Similar Matrices

Two matrices A & B are said to be similar if there exists a non-singular matrix L such that

$$B = L^{-1} A L$$

Sectular Equation or Characteristic Equation

The equation $|A - \lambda I| = 0$, where A is a square matrix is called the secular or characteristic equation of A.

[illegible] Similar matrices have same [illegible]r equation and hence the [illegible]e [illegible] values.

[illegible] Let A & B be similar matrices. Then ∃ a non-singular matrix L s.that

$$B = L^{-1} A L$$

Now secular equation for B is

$$|B - \lambda I| = 0$$

$$\Rightarrow |L^{-1} A L - \lambda I| = 0$$

$\because L^{-1} I L$
$\Rightarrow I L = I$

$$\Rightarrow [illegible] I L| = 0$$

[illegible] = 0

$$|L^{-1}|\,|AL - \lambda I|\,|L| = 0$$

$$|L^{-1}L|\,|AL - \lambda I| = 0$$

$$|I|\,|AL - \lambda I| = 0$$

$$\Rightarrow \quad |AL - \lambda I| = 0$$

$\Rightarrow$ Secular equations of A & B are same and also eigen values of A & B are same.

Theorem# If A is similar to a diagonal matrix D, then eigen values of A are equal to the diagonal elements of D.

Proof# Let A be similar to diagonal matrix D given by

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_2 \end{bmatrix}$$

The eigen values of D are given by

$$|D - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} d_1 - \lambda & 0 & 0 \\ 0 & d_2 - \lambda & 0 \\ 0 & 0 & d_3 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (d_1 - \lambda)(d_2 - \lambda)(d_3 - \lambda) = 0$$

$$\Rightarrow \lambda = d_1, d_2, d_3$$

$\because$ A is similar to D

$\therefore$ eigen values of A [illegible] d_3

Diagonalising Matrix

If A matrix L is such that

$$L^{-1} A L = D$$

when D is a diagonal matrix, then L is said to be diagonalising matrix

Method to find eigen Vector

If d_i are eigen values of a square matrix A. Then eigen vector corresponding d_j can be be found as

Let $\underline{c}_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{bmatrix}$ be eigen vector in column form.

Corresponding to eigen value d_j, then

$$(A - d_j I)\underline{c}_j = 0$$

from here we can find $c_{1,j}, c_{2,j} \ldots c_{nj}$ and hence $\underline{c}_j$.

Method to Diagonalise a Matrix

Let A be a square matrix which is to be diagonalised. To diagonalize it procede as

(1)# Find eigen values & corresponding eigen vectors for A and orthogonalise eigen vectors

(2) Form the matrix L, whose columns are orthogonalised eigen vectors of A. Then

$L^{-1} = L^t$ and Diagonalised form of A is given by

$$L^t A L = L^{-1} A L$$

Theorem # The eigen values of a symmetric matrix (Hermitian in complex space) are all real and if they are all distinct, then eigen vectors are orthogonal

Note # If A square matrix A is symmetric or Hermitian and if all of its eigen values are distinct, then its eigen vectors will be orthogonal and diagonalising matrix L as stated above can be made by simply putting eigen vectors of A in columns.

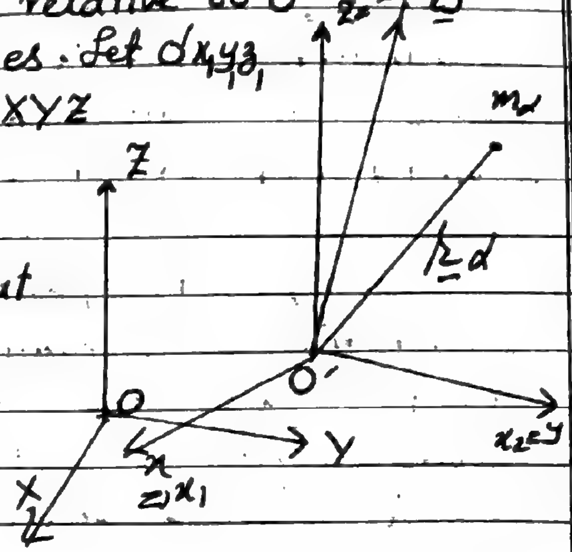
By Muhammad Hussain lecturer Govt. College Asghar Mall RWP.

Inertia Tensor

Problem # Define Inertia tensor, write it in dyadic form. Also prove that inertia tensor is a symmetric tensor of rank-two. Write inertia tensor in dyadic form. How many independent components intertia tensor have. How does the components of inertia tensor behave with the components of angular velocity. Write inertia tensor in case of continuous distribution of mass.

Sol # Inertia tensor is usually defined by the expression of angular momentum of rigid body relative a fixed point of body or relative to c.m of body which may be [illegible] rotating with body.

Consider a rigid body of n particles, rotating about a fixed point O' with instantaneous angular velocity $\underline{\omega}$. Let $\underline{r}_\alpha$ be p.v of α th particle w.r.t O'. Then $\underline{v}_\alpha = \dot{\underline{r}}_\alpha$ is velocity relative to O' and seen in fixed axes. Let $O x_1 y_1 z_1$ be body axes and $OXYZ$ be fixed axes. The angular momentum seen in fixed axes about O' is



$$\underline{L} = \sum_\alpha \underline{r}_\alpha \times m \underline{v}_\alpha$$

But $\underline{v}_\alpha = \underline{\omega} \times \underline{r}_\alpha$

$\Rightarrow$

$$\underline{L} = \sum_\alpha m_\alpha \underline{r}_\alpha \times (\underline{\omega} \times \underline{r}_\alpha)$$

$$= \sum_\alpha m_\alpha [\underline{r}_\alpha \cdot \underline{r}_\alpha \underline{\omega} - (\underline{r}_\alpha \cdot \underline{\omega}) \underline{r}_\alpha] \longrightarrow ①$$

$$= \sum_\alpha m_\alpha [r_\alpha^2 \underline{\omega} - (\underline{r}_\alpha \cdot \underline{\omega}) \underline{r}_\alpha] \longrightarrow ②$$

Let ω_i be components of $\underline{\omega}$, $r_{\alpha,i} = x_{\alpha,i}$ components $\underline{r}_\alpha$, L_i, components of $\underline{L}$. Also

$$\underline{\omega} \cdot \underline{r}_\alpha = \sum_{j=1}^{3} \omega_j r_{\alpha,j}$$

$$r_\alpha^2 = \sum_{k=1}^{3} r_{\alpha,k} r_{\alpha,k} = \sum_{k=1}^{3} x_{\alpha,k} x_{\alpha,k}$$

$$= \sum x_{\alpha,k}^2$$

So ith component of angular momentum is

$$L_i = \sum_\alpha m_\alpha [\omega_i \sum^3 x_{\alpha,k}^2 - x_{\alpha,i} \sum \omega_j x_{\alpha,j}]$$

putting $\omega_i = \sum_{j=1}^{3} \delta_{ij}\,\omega_j$

$$L_i = \sum_{\alpha} m_\alpha \left[\sum_{j=1}^{3} \delta_{ij}\omega_j \sum_{k=1}^{3} x_{\alpha,k}^{2} - x_{\alpha,i} \sum_{j=1}^{3} x_{\alpha,j}\,\omega_j \right]$$

$$= \sum_{j} \omega_j \sum_{\alpha} m_\alpha \left[\delta_{ij} \sum_{k=1}^{3} x_{\alpha,k}^{2} - x_{\alpha,i}\, x_{\alpha,j} \right]$$

$$\boxed{L_i = \sum_{j} I_{ij}\,\omega_j} \Longrightarrow (3)$$

where

$$I_{ij} = \sum_{\alpha} m_\alpha \left[\delta_{ij} \sum_{k=1}^{3} x_{\alpha,k}^{2} - x_{\alpha,i}\, x_{\alpha,j} \right] \longrightarrow (4)$$

is the ijth element of the inertia tensor. We denote inertia tensor by $\{I\}$ and its matrix form by $[I]$ given by

$$[I] = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

which is called inertia matrix. We know that
$I_{12} = I_{21}$ $\quad I_{13} = I_{31}$ $\quad I_{23} = I_{32}$.
Thus inertia tensor is symmetric (matrix) tensor and it will have six independent components, three diagonal elements, three off-diagonal elements.

Inertia Tensor for Continuous Distribution of mass

If we consider a body ~~(mass)~~ of continuous distribution of matter [illegible]

, then changing m_α by $dm = \rho\, dv$, where $dv = dx_1 dx_2 dx_3$ is the element of volume at point with position vector $\underline{r}$ or at point (x_1, x_2, x_3) so $\underline{r}_\alpha = \underline{r} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ and we drop α. Summation will be changed by integral symbol. ijth element from equation ① is

$$I_{ij} = \int_V \rho\left[\delta_{ij} \sum_{k=1}^{3} x_k^2 - x_i x_j\right] dv$$

$$\boxed{I_{ij} = \int_V \rho(\underline{r})\left[\delta_{ij}\, r^2 - x_i x_j\right] dv} \Rightarrow ⑤$$

where V is the volume of the body

Inertia Tensor in Dyadic form *

from ②

$$\underline{L} = \sum_\alpha m_\alpha\left[r_\alpha^2\, \underline{\omega} - \underline{r}_\alpha(\underline{r}_\alpha \cdot \underline{\omega})\right]$$

$$= \sum_\alpha m_\alpha\left[r_\alpha^2\, \underline{1}\cdot\underline{\omega} - \underline{r}_\alpha \underline{r}_\alpha \cdot \underline{\omega}\right]$$

$$= \sum_\alpha m_\alpha\left[r_\alpha^2\, \underline{1} - \underline{r}_\alpha \underline{r}_\alpha\right] \cdot \underline{\omega}$$

$$= \underline{I} \cdot \underline{\omega}$$

where $\underline{I} = \sum_\alpha m_\alpha\left[r_\alpha^2\, \underline{1} - \underline{r}_\alpha \underline{r}_\alpha\right] = \{I\}$

is moment of inertia in dyadic form or inertia tensor in dyadic form.

Inertia Tensor is of Rank Two *

~~[illegible]~~ two. This can be

from ③

$$L_i = \sum_j I_{ij} \omega_j$$

Since $\underline{\omega}$ is a vector (i.e. tensor of rank-one) and $\underline{L}$ is a vector, the inner-product $\sum_j I_{ij}\omega_j$ of quantity $\{I\}$ with $\underline{\omega}$ gives the components of a vector i.e tensor of rank one, therefor from quotient theorem I_{ij} are components of a tensor of rank two. Thus inertia tensor has rank two

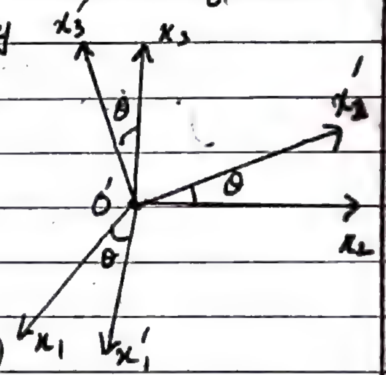
OR

Direct Method

The fundamental relation connecting inertia tensor and the angular (~~momentum~~) velocity is given by

$$L_k = \sum_l I_{kl} \omega_l \longrightarrow ⑥$$

This equation holds in body co-ordinate system $O'x_1, x_2, x_3$. Now suppose the body co-ordinate system is rotated within the body about O' in new position $O'x_1' x_2' x_3'$. Then in the rotated sysem



$$L_i' = \sum_j I_{ij}' \omega_j' \longrightarrow ⑦$$

Both $\underline{L}$ and $\underline{\omega}$ obey the standard transformation equation

$$x_i = \sum_j a_{ji} x_j'$$

Therefore we can write

$$L_k = \sum_m a_{mk} L_m' \longrightarrow ⑧$$

$$L_k = \sum_m a_{mk} L'_m \longrightarrow (8)$$

$$\omega_l = \sum_j a_{jl} \omega'_j \longrightarrow (9)$$

using (8) & (9) in (6)

$$\sum_m a_{mk} L'_m = \sum_l I_{kl} \sum_j a_{jl} \omega'_j$$

Multiplying both sides by a_{ik} and summing over k

$$\sum_m \left(\sum_k a_{ik} a_{mk} \right) L'_m = \sum_l I_{kl} \cdot \sum_k a_{ik} \sum_j a_{jl} \omega'_j$$

$$\sum_m (\delta_{im}) L'_m = \sum_j \left(\sum_{k,l} I_{kl} a_{ik} a_{jl} \right) \omega'_j$$

Summing over m

$$L'_i = \sum_j \left(\sum_{k,l} a_{ik} a_{jl} I_{kl} \right) \omega'_j \longrightarrow (10)$$

Comparing (7) & (10)

$$\sum_j I'_{jl} \omega'_j = \sum_j \left(\sum_{k,l} a_{ik} a_{jl} I_{kl} \right) \omega'_j$$

This is possible only if

$$I'_{jl} = \sum_{k,l} a_{ik} a_{jl} I_{kl} \longrightarrow (11)$$

which is a rule for the transformation of the components of a 2nd rank tensor. Then inertia tensor $\{I\}$ is a 2nd rank Tensor.

Note From (11)

$$I'_{jl} = \sum a_{ik} I_{kl} a^t_{lj}$$

If $[I]$ & $[I']$ are inertia matrices into un-primed and primed system and $A = [a_{ij}]$ is matrix of transformation, then above equation in matrix form is

$$[I'] = A[I]A^t$$

$\because$ A is orthogonal transformation matrix

$\therefore$ $A^t = A^{-1}$

$$\Rightarrow [I'] = A[I]A^{-1}$$

A transformation of this type is similarity transformation & $[I']$ is similar to $[I]$

Behaviour of Components of inertia tensor with Components of angular Velocity # & Behaviour of inertia matrix with angular velocity vector.

We have

$$L_i = \sum_j I_{ij}\omega_j$$

$$L_1 = \sum_j I_{1j}\omega_j = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \longrightarrow ⓐ$$

$$L_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \longrightarrow ⓑ$$

$$L_3 = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \longrightarrow ⓒ$$

writing these in matrix form

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$[\underline{L}] = [I][\underline{\omega}]$$

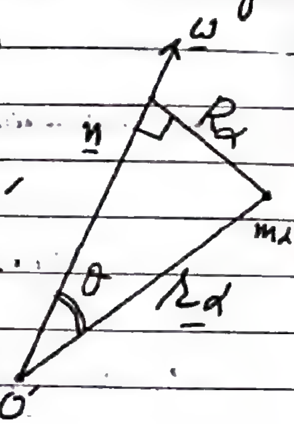
Equations ⓐ ⓑ ⓒ ... for each component of

angular momentum is a linear combination of all the components of angular velocity. The numbers connecting linearly the components of angular velocity to give a component of angular momentum are sine elements of inertia matrix.

Eq (d) shows that the angular momentum vector is related to the angular velocity by a linear transformation.

Remarks # We can also define the inertia tensor from the expression of rotational K.E.

Also we can define inertia tensor from the expression of moment of inertia about an instanteous axis through a stationary point of the body as



Let $\underline{n} = [n_1, n_2, n_3]$ be unit vector along the instantaneous axis of rotation through the fixed point O′ and $\underline{r}_\alpha$ be p.v of αth particle. R_α is $\perp$ar distance of m_α from the axis. The ~~angular~~ moment of inetia about axis is

$$\therefore I = \sum_\alpha m_\alpha R_\alpha^2 = \sum m_\alpha (\underline{r}_\alpha \times \underline{n})^2$$

$$= \sum_\alpha m_\alpha (\underline{n} \times \underline{r}_\alpha) \cdot (\underline{n} \times \underline{r}_\alpha) \rightarrow ①$$

Now $(\underline{n} \times \underline{r}_\alpha)_k = \epsilon_{ijk} n_i r_{\alpha,i}$

$$= \epsilon_{ijk} n_i x_{\alpha,j}$$

$$(\underline{n} \times \underline{r}_\alpha) \cdot (\underline{n} \times \underline{r}_\alpha) = (\underline{n} \times \underline{r}_\alpha)_k (\underline{n} \times \underline{r}_\alpha)_k$$

$$= \epsilon_{ijk} n_i x_{\alpha,j} \, \epsilon_{lmk} n_l x_{\alpha,m}$$

using in ①

$$I = \sum_{\alpha} m_\alpha \, \epsilon_{ijk} \epsilon_{lmn} n_i x_{\alpha,j} n_l x_{\alpha,m}$$

i, j, k, l, m in R.H.S are dummies

$$\epsilon_{ijk} \epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$I = \sum_{\alpha} m_\alpha \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \right] n_i n_l x_{\alpha,j} x_{\alpha,m}$$

$$= \sum_{\alpha} m_\alpha \delta_{il}\delta_{jm} n_i n_l x_{\alpha,j} x_{\alpha,m} - \sum_{\alpha} m_\alpha \delta_{im}\delta_{jl} n_i n_l x_{\alpha,j} x_{\alpha,m}$$

$$= \sum_{\alpha} m_\alpha (n_i n_i x_{\alpha,j} x_{\alpha,j} - n_i n_j x_{\alpha,i} x_{\alpha,j})$$

$$= \sum m_\alpha (n_i n_i r_\alpha^2 - n_i n_j x_{\alpha,i} x_{\alpha,j})$$

putting $n_i = \sum_j \delta_{ij} n_j = \delta_{ij} n_j$

$$I = \sum m_\alpha \left[n_i n_j \delta_{ij} r_\alpha^2 - n_i n_j x_{\alpha,i} x_{\alpha,j} \right]$$

$$= \sum m_\alpha n_i n_j \left[\delta_{ij} r_\alpha^2 - x_{\alpha,i} x_{\alpha,j} \right]$$

$$= \sum_{\alpha} m_\alpha \left[\delta_{ij} r_\alpha^2 - x_{\alpha,i} x_{\alpha,j} \right] n_i n_j$$

$$I = I_{ij} n_i n_j$$

where $I_{ij} = \sum_{\alpha} m_\alpha \left[\delta_{ij} r_\alpha^2 - x_{\alpha,i} x_{\alpha,j} \right]$. is

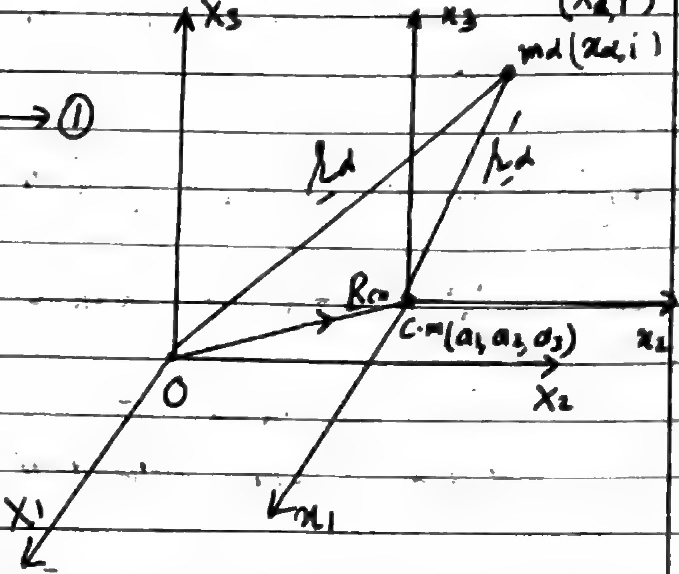
ijth element of inertia tensor. Hence inertia tensor $[I]$ will be completely defined by this

Generalised Parallel Axes Theorem
OR
Parallel Axes Theorem for Components of Inertia Tensor:
OR
Moment of Inertia for different Body Co-ordinate System

Theorem# Discussing the significance, of the parallel axes theorem, state and prove it both in tensorial form and dyadic form

Proof# K.E of rigid body can be seperated into rotational and translational parts only if the origin of the body Co-ordinate system is taken at C.m. For certain geometrical shapes, it may not always be convenient to compute the elements of inertia tensor using such a Co-ordinate System. Therefor consider some other set of Co-ordinate axes X_i fixed with respect to body at poin O of the body. Let x_i be body axes with origin at C.m C whose Co-ordinates relative to O are (a_1, a_2, a_3). Let p.v of C.m w.r.t O be $\underline{R}_{cm}$ and p.vs from O and centre of mass to the αth particle be $\underline{r}_\alpha$, $\underline{r}'_\alpha$ respectively (Fig).

Suppose X_i, x_i axes have same orientation i.e are parallel respectively. Note O may be located either within or outside of the body under consideration. Now



$$\underline{r}_\alpha = \underline{R}_{cm} + \underline{r}'_\alpha \longrightarrow ①$$

The elements of inertia tensor relative to X_i - axes are

$$I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} \sum_{k=1}^{3} X_{\alpha,k}^2 - X_{\alpha,i} X_{\alpha,j}\right] \longrightarrow ②$$

from ① Components are

$$X_\alpha = a_k + x_\alpha \qquad k = 1,2,3$$
$$\alpha = 1,2,3$$

$$X_{\alpha,k} = a_k + x_{\alpha,k}$$

Then tensor element I_{ij} becomes

$$I_{ij} = \sum m_\alpha \left[\delta_{ij} \sum_{k=1}^{3} (x_{\alpha,k} + a_k)^2 - (x_{\alpha,i} + a_i)(x_{\alpha,j} + a_j)\right]$$

$$= \sum m_\alpha \left[\delta_{ij} \sum_{k=1}^{3} \left(x_{\alpha,k}^2 + 2 x_{\alpha,k} a_k + a_k^2\right) - (a_i x_{\alpha,j} + a_j x_{\alpha,i} + x_{\alpha,i} x_{\alpha,j}\right.$$

$$= \sum m_\alpha \left[\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right] + \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k (2 x_{\alpha,k} a_k + a_k^2)\right]$$

$$- \sum_\alpha m_\alpha (a_i x_{\alpha,j} \ldots$$

$$I_{ij} = \sum_\alpha m_\alpha\left[\delta_{ij}\sum_k x_{\alpha,k}^2 - x_{\alpha,i}x_{\alpha,j}\right] + \sum_\alpha m_\alpha\left[\delta_{ij}\sum_k a_k^2 - a_i a_j\right]$$
$$+ \sum_\alpha m_\alpha\left[2\delta_{ij}\sum_k x_{\alpha,k}a_k - a_i x_{\alpha,j} - a_j x_{\alpha,i}\right]$$

$$I_{ij} = I'_{ij} + \sum_\alpha m_\alpha\left[\delta_{ij}\sum_k a_k^2 - a_i a_j\right] + \sum_\alpha m_\alpha\left[2\delta_{ij}\sum x_{\alpha,k}a_k - a_i x_{\alpha,j} - a_j x_{\alpha,i}\right]$$

$\because$ $\underline{r}'_\alpha$ is relative to c.m

$$\therefore \quad \sum_\alpha m_\alpha \underline{r}'_\alpha = 0$$

for the kth component

$$\sum_\alpha m_\alpha x_{\alpha,k} = 0$$

and

$$\sum_\alpha m_\alpha x_{\alpha,j} = 0 = \sum_\alpha m_\alpha x_{\alpha,i}$$

$\rightarrow$

$$I_{ij} = I'_{ij} + \sum m_\alpha\left[\delta_{ij}\sum_k a_k^2 - a_i a_j\right]$$

But $\sum_\alpha m_\alpha = M =$ total mass of the body

$$\sum_k a_k^2 = R_{cm}^2$$

$$I_{ij} = I'_{ij} + M\left[\delta_{ij}R_{cm}^2 - a_i a_j\right]$$

$$\boxed{I'_{ij} = I_{ij} - M\left[\delta_{ij}R_{cm}^2 - a_i a_j\right]} \Longrightarrow (3)$$

Where I'_{ij} ijth element of inertia tensor relative to co-ordinate axes at c.m and $M[\delta_{ij}R_{cm}^2 - a_i a_j]$ is inertia tensor referred to point O for a point mass M [illegible] we can calculate elements

I'_{ij} of the desired inertia tensor once Those w.r.t the X_i – axes are known & vice versa

Equation ③ is the general form of the Steiner's parallel axes theorem

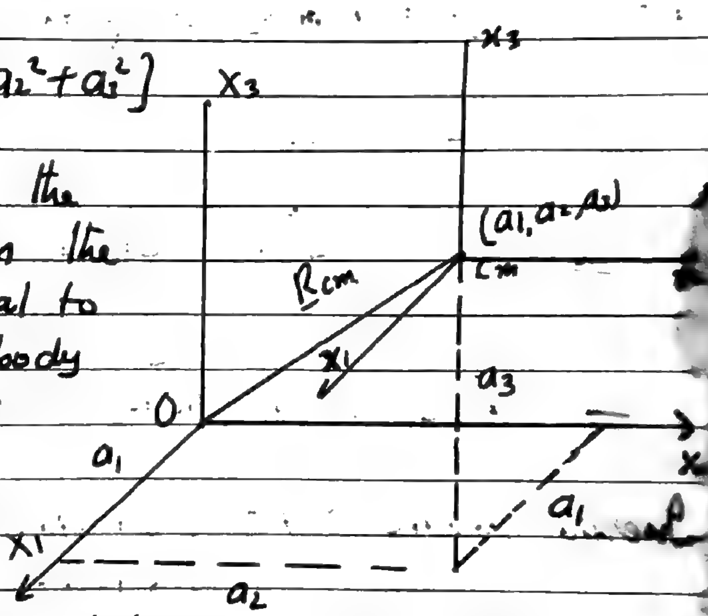
For $i=j$ equation ③ gives moment of inertia about x_i-axes & X_i-axes and for $i \neq j$ equation ③ gives products of inertia relative to x_i-axes & X_i-axis.

$$I'_{ii} = I_{ii} - M[R_{cm}^2 - a_i^2]$$

$$I'_{11} = I_{11} - M[a_1^2 + a_2^2 + a_3^2 - a_1^2]$$

$$= I_{11} - M[a_2^2 + a_3^2]$$

$$\Rightarrow I_{11} - I'_{11} = M[a_2^2 + a_3^2]$$



which states that the difference between the elements is equal to the mass of the body multiplied by the square of the distance between the parallel axes (in this case between x_1 & X_1-axis)

In Dyadic Form

Moment of inertia relative to o is given by

$$I_O = \sum_\alpha m_\alpha [r_\alpha^2 \underline{1} - \underline{r}_\alpha \ldots$$

from ①

$$\underline{\underline{I}}_O = \sum_\alpha m_\alpha \left[(\underline{R}_{cm} + \underline{r}'_\alpha)^2 \underline{\underline{1}} - (\underline{R}_{cm} + \underline{r}'_\alpha)(\underline{R}_{cm} + \underline{r}'_\alpha) \right]$$

$$= \sum_\alpha m_\alpha \left[(\underline{R}_{cm} + \underline{r}'_\alpha)\cdot(\underline{R}_{cm} + \underline{r}'_\alpha)\underline{\underline{1}} - \underline{R}_{cm}\underline{R}_{cm} - \underline{R}_{cm}\underline{r}'_\alpha - \underline{r}'_\alpha \underline{R}_{cm} - \underline{r}'_\alpha \underline{r}'_\alpha \right]$$

$$= \sum_\alpha m_\alpha \left[(R_{cm}^2 + 2\underline{R}_{cm}\cdot\underline{r}'_\alpha + \underline{r}'_\alpha\cdot\underline{r}'_\alpha)\underline{\underline{1}} - \underline{R}_{cm}\underline{R}_{cm} - \underline{R}_{cm}\underline{r}'_\alpha - \underline{r}'_\alpha \underline{R}_{cm} - \underline{r}'_\alpha \underline{r}'_\alpha \right]$$

$$\because \sum_\alpha m_\alpha \underline{r}'_\alpha = \underline{0}$$

$\Rightarrow$

$$\underline{\underline{I}}_O = \sum_\alpha m_\alpha R_{cm}^2 \underline{\underline{1}} + \sum_\alpha m_\alpha r'^2_\alpha \underline{\underline{1}} - \sum_\alpha m_\alpha \underline{R}_{cm}\underline{R}_{cm} - \sum_\alpha \underline{r}'_\alpha \underline{r}'_\alpha$$

$$= \sum_\alpha m_\alpha (r'^2_\alpha \underline{\underline{1}} - \underline{r}'_\alpha \underline{r}'_\alpha) + M(R_{cm}^2 \underline{\underline{1}} - \underline{R}_{cm}\underline{R}_{cm})$$

$$= \underline{\underline{I}}_{cm} + M(R_{cm}^2 \underline{\underline{1}} - \underline{R}_{cm}\underline{R}_{cm})$$

Inertia diadic w.r.t O = Inertia diadic w.r.t cm + Inertia diadic of Mass M at centre of mass w.r.t O

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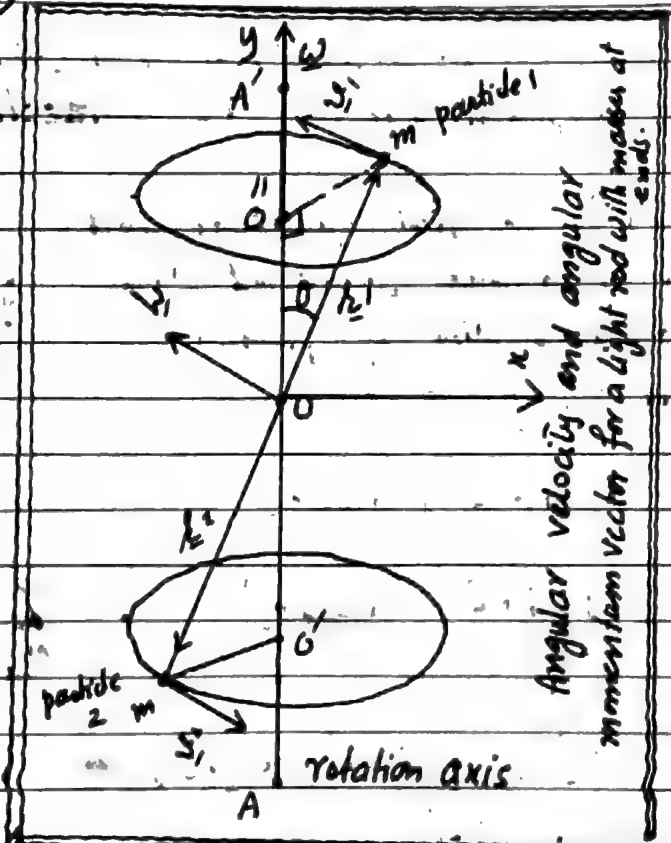
Rotation about Fixed Axes and Behaviour of the angular Momentum Vector

In general the angular velocity vector ω and angular momentum vector L are not parallel or collinear. We illustrate this by a simple example of rigid body consisting of two equal mass points joined by a massless rod which rotates about fixed through centre of mass and makes angle θ with the rod. (Such equal masses at ends of massless rod form a dumb-bell)



Angular velocity and angular momentum vector for a light rod with masses at ends.

Consider the dumb-bell in the plane of paper in xy-plane. Length of rod is 2a and the angular velocity [illegible] in fixed [illegible] and the axes are [illegible]

If $\underline{r}_1$, $\underline{r}_2$ are p.vs of particles and $\underline{L}_1$, $\underline{L}_2$ are their angular momentum about O, then $\underline{L}_1$ & $\underline{L}_2$ both point in the same direction which is $\perp$ar to rod ~~and out of the plane of~~ Paper ~~at the instant the rod coincides with xy-plane~~.

We have pictured the system when the rod coincides with xy-plane

$$\underline{v}_1 = \underline{\omega} \times \underline{r}_1 \qquad \underline{v}_2 = \underline{\omega} \times \underline{r}_2 \ldots$$

i.e. $\underline{v}_\alpha = \underline{\omega} \times \underline{r}_\alpha$

The instantaneous angular momentum of the system is

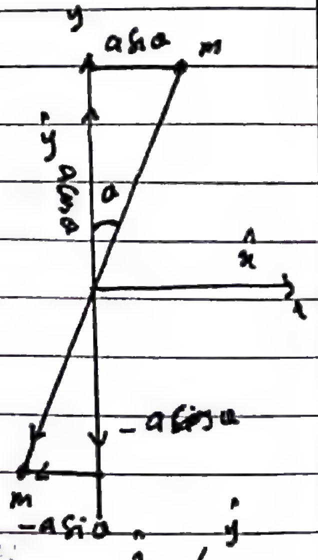
$$\underline{L} = \underline{L}_1 + \underline{L}_2$$

$$= \sum_\alpha m_\alpha \underline{r}_\alpha \times \underline{v}_\alpha = \sum_\alpha m_\alpha \underline{r}_\alpha \times (\underline{\omega} \times \underline{r}_\alpha)$$

$$= m \underline{r}_1 \times (\underline{\omega} \times \underline{r}_1) + m (\underline{\omega} \times \underline{r}_2)$$

$\underline{L}$ is perpendicular to line connecting the masses

particle 1 is in first quadrant and particle 2 is in 3nd quadrant when rod coincides with xy-plane.

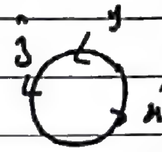


$$|\underline{r}_1| = |\underline{r}_2| = a$$

$$\underline{r}_1 = a\sin\theta\,\hat{x} + a\cos\theta\,\hat{y}$$

$$\underline{r}_2 = -a\sin\theta\,\hat{x} - a\cos\theta\,\hat{y}$$

$$\underline{\omega} = \omega\hat{y}$$



$$\underline{\omega} \times \underline{r}_1 = \omega\hat{y} \times (a\sin\theta\,\hat{x} + a\cos\theta\,\hat{y})$$

$$= -\omega a\sin\theta\,\hat{z}$$

$$\underline{r}_1 \times \underline{\omega} \times \underline{r}_1 = (a\sin\theta\,\hat{x} + a\cos\theta\,\hat{y}) \times -\omega a\sin\theta\,\hat{z}$$

[illegible]

[illegible] $\hat{y}$ $-$ [illegible] $a^2\sin\theta\cos\theta\,\hat{x}$

$m\underline{r}_1$ [illegible] $-\sin\theta\cos\theta\,\hat{x})$

$$m\underline{r_1}\times(\underline{\omega}\times\underline{r_1}) = ma^2\omega^2\sin\theta(\sin\theta\,\hat{y} - \cos\theta\,\hat{x}) \rightarrow (1)$$

$$\underline{\omega}\times\underline{r_2} = \omega\hat{y}\times(-a\sin\theta\,\hat{x} - a\cos\theta\,\hat{y})$$

$$= \omega a\sin\theta\,\hat{z}$$

$$m\underline{r_2}\times(\underline{\omega}\times\underline{r_2}) = m(-a\sin\theta\,\hat{x} - a\cos\theta\,\hat{y})\times\omega a\sin\theta\,\hat{z}$$

$$= m\omega a^2\sin^2\theta\,\hat{y} - m\omega a^2\sin\theta\cos\theta\,\hat{x}$$

$$= m\omega a^2\sin\theta(\sin\theta\,\hat{y} - \cos\theta\,\hat{x}) \longrightarrow (2)$$

Adding (1) & (2)

$$\underline{L} = 2m\omega a^2\sin\theta(\sin\theta\,\hat{y} - \cos\theta\,\hat{x}) \longrightarrow (3)$$

If $\hat{a}$ is unit vector along rod, then

$$\hat{a} = \cos\theta\,\hat{y} + \sin\theta\,\hat{x}$$

Let $\hat{L} = \sin\theta\,\hat{y} - \cos\theta\,\hat{x}$

$$\hat{a}\cdot\hat{L} = \sin\theta\cos\theta - \sin\theta\cos\theta = 0$$

$\Rightarrow$ $\underline{L}$ is perpendicular to the rod and $\underline{\omega}$ and directed along the axis of rotation. This fact also can be seen that $\underline{L}$ is not constant (scalar) multiple of $\underline{\omega}$

$$|\underline{L}| = 2m\omega a^2\sin\theta$$

We note that the angular momentum vector $\underline{L}$ does not remain constant in time, but rotates with an angular velocity $\underline{\omega}$ in such a way it traces out a cone whose axis is the axis of rotation.

But in fact

$$\frac{d\underline{L}}{dt} = \underline{\omega} \times \underline{L}$$ (By derivative w.r.t rotating axis)

$$= -2m\omega^2 a^2 \sin\theta \cos\theta \, \hat{z}$$

where $\hat{z}$ points out of the plane of diagram.

But if $\frac{dL}{dt} \neq 0$, then there must exist a torque giving rise to changing angular momentum vector. So

$$\underline{N} = \frac{dL}{dt} = -2\omega^2 a^2 m \sin\theta \cos\theta \, \hat{z}$$

This torque vector also rotates with the rod as does $\underline{L}$ and is a vector denoting the direction in which the end of the vector $\underline{L}$ is moving.

The rotating torque must be supplied by the bearings (not shown) that hold the rod and constrain it to rotate at angle θ about the y-axis.

Reason for this torque can seen as

$$\underline{v}_1 = \underline{\omega} \times \underline{r}_1 = -\omega a \sin\theta \, \hat{z}$$

$$v_1 = \omega a \sin\theta$$

Radius of circular path of particle $= r_1 \sin\theta$

$$= a\sin\theta = R_1$$

$$F_c = \frac{m v_1^2}{R_1} = \frac{m\omega^2 a^2 \sin^2\theta}{a\sin\theta} = m\omega^2 a^2 \sin\theta$$

$$F_1 = m\omega^2 a^2 \sin\theta$$

These two forces are opposite in direction and $\perp$ to axis of rotatation at the centre of their circular paths.

Distance b/w forces $= O'O'' = OO' + OO''$
$= a\cos\alpha + a\cos\alpha = 2a\cos\alpha$

Moment of this couple $= N = (2a\cos\alpha)(m\omega^2 a^2 \sin\alpha)$

This torque is supplied to particle via the rigid rod from bearing at A & A'.

By Newton's 2nd Law ~~on~~ the torque which the rotating dumb-bell exerts on the bearings (say at A and A') is equal and opposite to this.

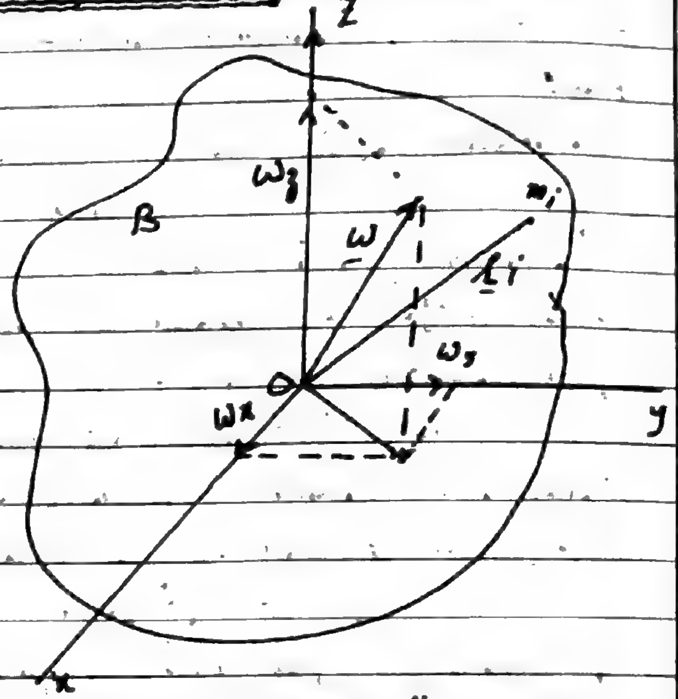
If $\theta = 90°$, then ω is along the line of Symmetry for this simple mass distribution and $\underline{L}$ coincides with direction of $\underline{\omega}$. Then if $\underline{\omega}$ is Constant, the so is $\underline{L}$ and no rotating torque is required to satisfy the motion.

The opposite forces $F_{c_1} = m\omega^2 a^2 \sin\theta$ & $F_{c_2} = m\omega^2 a^2 \sin\theta$ cause a wobble in the upper and lower bearings. For the Symmetrical rotating body there is no bearing wobble and the shaft rotates Smoothly.

Remarks # We have noted above that in general angular momentum vector $\underline{L}$ and angular velocity vector are not parallel. If we consider a rigid body for which which the inertia tensor $\{I\}$ has non-vanishing off diagonal elements, then even if $\underline{\omega}$ is directed along, say x_1-axis, $\underline{\omega} = [\omega_1, 0, 0]$, the angular momentum vector will be general have nonvanishing Components in all Three direction: $\underline{L} = [L_1, L_2, L_3]$ [illegible]

[illegible]

Importance of Principlal Axes



Consider a rigid body B which is rotating at a certain instan with angular velocity $\underline{\omega}$ about an axis passing through a given reference point O which is attached to the body (it need not lie with the body)

Suppose oxyz is non rotating and the origin O is such that $\dot{L} = N$ i.e origin O is fixed point or c.m or $\ddot{\underline{r}}_o$ (acc of O relative some fixed pt) is directed through c.m.

$$\underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

The quantities $\omega_x, \omega_y, \omega_z$ will be function of time if relative to Oxyz, $\underline{\omega}$ is changing in magnitude or direction or both

Angular momentum relative to origin O is

$$\underline{L} = \sum_{i=1}^{n} m_i \underline{r}_i \times (\underline{\omega} \times \underline{r}_i)$$

$$\underline{L} = (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\hat{i} + \hat{j}(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)$$

$$+ \hat{k}(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)$$

It is clear that if the co-ordinates axes are not rigidly attached to the body and rotating with it, the moments and products of inertia will be in general function of time. For this reason it is convenient to employ such co-ordinate

systems which are so attached to body that the moments and products of inertia w.r.t these are constant. This means that the axes must be performing some or all of the rotational motions described by the body, the extent to which the axes follow the motions of the body being dependent upon the degree of symmetry possessed by the body.

A particular set of such axes is especially useful. This set is such that all the products of inertia are zero. A set of axes possessing this property is called a set of principal axes at the point o. With reference to such axes the expressions of K.E and angular momentum are simplified to forms which are easy to use and Further calculations are also simplified

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Principal Axes

A set of body axes for which the products of inertia (i.e the off-diagonal elements of {I}) vanish are called the principal axes of of inertia or body. The origin of these axes is called principal point, Co-ordinate planes are called principal planes and the moments of inertia about these principal axes are called principal moments of inertia

Simplification of Expressions for kinetic Energy angular momentum and Components of Inertia Tensor relative to Principal Axes

The expressions for K.E of rigid body, for the Components of angular momentum about a fixed point of body or about c.m of body which may translating are given relative to body axes as

$$T_{rot} = \frac{1}{2} \sum_{i,j} I_{ij}\, \omega_i \omega_j \longrightarrow (1)$$

$$L_i = \sum_j I_{ij}\, \omega_j \longrightarrow (2)$$

where I_{ij} are Components of inertia tensor $\{I\}$.

Now if the body axes are priciptal axes, then

$$I_{ij} = 0 \qquad \text{for } i \neq j$$

$$= I_{jj} = I_j \qquad \text{for } i = j$$

and we can write these Compments as

$$I_{ij} = \sum_i \delta_{ij} I_i = \delta_{ij} I_i$$

Then the inertia tensor would simplify to

$$\{I\} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

and expressions (1) & (2) would simplify as

$$T_{rot} = \frac{1}{2} \sum_{i,j} I_i \delta_{ij}\, \omega_i \omega_j$$

$$= \frac{1}{2} \sum \sum I_i \delta_{ij}\, \omega_i \omega_j$$

$$T_{rot} = \frac{1}{2}\sum_{i} I_i \omega_i \omega_i$$

$$\boxed{T_{rot} = \frac{1}{2}\sum_{i} I_i \omega_i^2} \longrightarrow (3)$$

which can be written in expanded form as

$$\boxed{T_{rot} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)}$$

OR

$$\boxed{T_{rot} = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right)}$$

$$L_i = \sum_{j}\sum_{j} I_i \delta_{ij}\omega_j$$

$$= \sum_{j} I_i \delta_{ij}\omega_j$$

$$L_i = I_i\omega_i$$

$$\boxed{L_i = I_i\omega_i} \longrightarrow (4)$$

i is not summation index here i.e not dummy

$$\Rightarrow L_1 = I_1\omega_1$$
$$L_2 = I_2\omega_2$$
$$L_3 = I_3\omega_3$$
$$\underline{L} = I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k}$$

Remarks # Since product inertia are zero i.e

$$I_{12} = I_{13} = I_{23} = I_{21} = I_{31} = I_{32} = 0$$

, therefore we use single subsript to denote the moments of inertia about principal axes. i.e I_{12} I_{11} or $I_x = I_{xx}$ etc because double subscripts are used only to maintain the symmetry of notation with that for product of inertia which are now absent and so no question of such symmetry.

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Remarks # The principal axes are fixed relative to the body and for this reason they do not in general form an inertial frame. In fact they rotate with the body or at least they maintain a relation ship to it such that the inertial properties of the body are constant when referred to these axes.

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Behaviour of Angular Momentum and angular Velocity Vector when Co-ordinate Axes are Principal Axes

Problem # Find the Conditions under which angular momentum vector and velocity vector are parallel, when Co-ordinate axes are chosen as principal axes #

Sol # Suppose the Co-ordinate axes are

are chosen as principal axes and a rigid body rotates with an angular velocity

$$\underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \longrightarrow (1)$$

Since Co-ordinate axes are principal axes, therefore angular momentum $\underline{L}$ is.

$$\underline{L} = I_1 \omega_x \hat{i} + I_2 \omega_y \hat{j} + I_3 \omega_z \hat{k} \longrightarrow (2)$$

$$L_1 = I_1 \omega_x \qquad L_2 = I_2 \omega_y \qquad L_3 = I_3 \omega_z$$

$\underline{L}$ and $\underline{\omega}$ are not in general parallel or in the same direction. These vectors when expressed relative to principal axes may have same direction under the following two Conditions

<u>Condition # 1</u> # If $I_1 = I_2 = I_3$, then from (2)

$$\underline{L} = I_1 \omega_x \hat{i} + I_1 \omega_y \hat{j} + I_1 \omega_z \hat{k}$$

$$= I_1 (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k})$$

$$\underline{L} = I_1 \underline{\omega}$$

$\Rightarrow$ $\underline{L}$ is scalar multiple of $\underline{\omega}$ or ratios of the Components of two vectors are same

$\Rightarrow$ $\underline{L}$ & $\underline{\omega}$ have same direction and Collinear.

<u>Condition 2</u> # If the body rotates about one of principal axes say x-axis, then $\omega_y = \omega_z = 0$ and

$$\underline{L} = I_1 \omega_x \hat{i} = I_1 \underline{\omega}$$

$\rightarrow$ $\underline{L}$ & $\underline{\omega}$ are collinear & have same direction.

Existence of Principal Axes

An Explaination # Before we prove the existence of the pricipal axes for a rigid it seems necessary to give an explatination about inertia matrix or matrix of the components of inertia tensor

When refererred to principal axes, the inertia tensor will consist of only diagonal elements and we can write its components as

$$I_{ij} = I_i \delta_{ij}$$

and inertia tensor {I} would be

$$\{I\} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \longrightarrow (1)$$

and inertia matrix [I] would be

$$\{I\} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \longrightarrow (2)$$

It means that the removal of non-diagonal elements i.e diagonalisation of inertia tensor or inertia matrix, is to find a set of pricipal axes. Now we know that diagonalisation of a matrix amounts to finding

eigen values and eigen vectors. We also know that that the eigen values ~~and eigen vectors~~ of a symmetric matrix (Hermitian in complex space) are real and its eigen vectors are orthogonal if eigen values are all distinct. Since inertia matrix [I] is real symmetric matrix, it eigen values will be real and eigen vectors orthogonal if all the eigen values unequal (distinct). The orthogonal vectors will serve as a set of principal axes and the eigen values will be required diagonal components. Hence the problem of finding a set of axes in which [I] is diagonal is equivalent to the eigen value problem for the matrix [I] or tensor {I}.

From this explanation we come to know that we can prove the existence of principal axes by following ~~the~~ methods

(1) by diagonalizing inertia matrix [I] i.e by finding eigen values and eigen vectors.

(2) by digonalizing inertia tensor

We can also find principal axes by geometric method with the help of momental ellipsoid which will be discussed later on

To prove the existence of principal axes, we state and prove the Theorem on next page.

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Theorem # For a rigid body there exists a set of three mutually orthogonal axes, called principal axes relative to which the product of inertia are zero and $\underline{L}$, $\underline{\omega}$ are oriented along the same direction.

OR

How will you find principal axes if there exists one such axis.

Proof # Suppose a rigid body rotates about a principal axis through a fixed point O or about c.m (in this case body may be general motion) with an instantaneous angular (~~Momentum~~) velocity $\underline{\omega}$. Then angular momentum $\underline{L}$ and angular ~~mom~~ velocity $\underline{\omega}$ are directed along this axis and we can write

$$\underline{L} = \lambda \underline{\omega} \longrightarrow (1)$$

$$\Rightarrow L_1 = \lambda\omega_1 \qquad L_2 = \lambda\omega_2, \quad L_3 = \lambda\omega_3$$

Also we have

$$[\underline{L}] = [I][\underline{\omega}] \longrightarrow (2)$$

$$\text{where } [\underline{L}] = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} \lambda\omega_1 \\ \lambda\omega_2 \\ \lambda\omega_3 \end{bmatrix}$$

So from (2)

$$\begin{bmatrix} \lambda\omega_1 \\ \lambda\omega_2 \\ \lambda\omega_3 \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} \lambda\omega_1 \\ \lambda\omega_2 \\ \lambda\omega_3 \end{bmatrix} = \begin{bmatrix} I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ I_{21}\omega_1 + I_{22}\omega_2 \quad I_{23}\omega_3 \\ I_{31}\omega_1 + I_{32}\omega_2 \quad I_{33}\omega_3 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} \lambda\omega_1 &= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ \lambda\omega_2 &= I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ \lambda\omega_3 &= I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned}\right\} \rightarrow (3)$$

$$\Rightarrow \left.\begin{aligned} (I_{11}-\lambda)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \\ I_{21}\omega_1 + (I_{22}-\lambda)\omega_2 + I_{23}\omega_3 &= 0 \\ I_{31}\omega_1 + I_{32}\omega_2 + (I_{33}-\lambda)\omega_3 &= 0 \end{aligned}\right\} \rightarrow (4)$$

The system of equations in (4) will have non-trivial solution if det of the co-efficient vanish i.e if

$$\begin{vmatrix} I_{11}-\lambda & I_{12} & I_{13} \\ I_{21} & I_{22}-\lambda & I_{23} \\ I_{31} & I_{32} & I_{33}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left| \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| [I] - \lambda I \right| = 0 \longrightarrow (5)$$

which is called secular equation

where $[I]$ = inertia matrix
I = Identity matrix

Equation (5) is called secular Equation or characteric equation of inertia matrix because from (1) & (2), we have

$$[\lambda \underline{\omega}] = [I][\underline{\omega}]$$

$$[I][\underline{\omega}] = \lambda[\underline{\omega}]$$

where $[\underline{\omega}] = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ Colum vector

So $[\underline{\omega}]$ is eigen vector and λ is an eigen ~~vector~~ value correspondig to eigen vector $[\underline{\omega}]$.

Equation (5) is cubic in λ and will have three roots or three eigenvalues say $\lambda_1, \lambda_2, \lambda_3$. Since the inertia matrix is symmetric, there all the eigen values will be real. Also all the three roots will +ve because if the root is -ve, then $\underline{L}$ & $\underline{\omega}$ as seen from (1) will be in opposite direction which is not the case here. Now following case for $\lambda_1, \lambda_2, \lambda_3$ may arise

Case I # If the eigen values are all distinct, then the eigen values will be orthogonal to each other and these will form, the system of principal axes.

Case II If two or three eigen values are equal or degenerate, then mutually orthogonal vectors can still be found (By Gram-Schmidt process or otherwise)

The direction cosines of a principal axes can be found directly from equations (4)

Since for these equations the directions of $\underline{\omega}$ and the pricipal axes coincide we have for a principal axis with D.Cs l, m, n

$$l = \frac{\omega_1}{\omega} \qquad m = \frac{\omega_2}{\omega} \qquad n = \frac{\omega_3}{\omega}$$

$$\Rightarrow \omega_1 = l\omega \qquad \omega_2 = m\omega \qquad \omega_3 = n\omega$$

using these in ④, we get

$$\left.\begin{aligned} (I_{11} - \lambda) l &+ I_{12} m + I_{13} n = 0 \\ I_{21} l &+ (I_{22} - \lambda) m + I_{23} n = 0 \\ I_{31} l &+ I_{32} m + (I_{33} - \lambda) n = 0 \\ \text{Als} \quad & \\ l^2 + m^2 + n^2 &= 1 \end{aligned}\right\} \rightarrow ⑥$$

Equations in ⑥ enable a solution for the direction cosines for each of three λ,s seperately. The transformation of inertia terms from a set of centroidal axes to a parallel set of non-centroidal axes may be handled by parallel axes Theorem.

Method-II to find D.Cs of Principal axes

Since the direction of $\underline{\omega}$ w.r.t body ~~principal~~ axis will be same as the direction of the principal axis corresponding to an eigen value say λ_1 ($\underline{L} = \lambda_1 \underline{\omega}$), therefor we can determine the direction of this principal axis by putting λ_1 for λ in equations ④, and finding the ratios of the components of the angular velocity vector $\omega_1 : \omega_2 : \omega_3$ and then find the direction cosines of corresponding principal axis. The directions corresponding to λ_2, λ_3 can be found similarly.

Note # The fact that above procedure yields only the ratios of the components of ω is no

handicap, since the ratio completely determine the direction of each principal axes and it is only the (principal) directions of principal axes that is required. Ideed we would not require the magnitudes of the ω_i, since the actual rate of angular motion canot be specified by geometry alone; we are free to impress (give) on the body any magnitude of the angular velocity that we wish.

We can now prove that product of inertia about the axes corresponding to eigen values $\lambda_1, \lambda_2, \lambda_3$ are zero and m.o.i about these axes are $I_{11} = \lambda_1$ $I_{22} = \lambda_2$ $I_{33} = \lambda_3$

The angular momentum $\underline{L}$ is

$$\underline{L} = \sum_i m_i r_i^2 \underline{\omega} - \sum_i m_i (\underline{r}_i \cdot \underline{\omega}) \underline{r}_i \longrightarrow (7)$$

But from (1) $\underline{L} = \lambda \underline{\omega}$

$\Rightarrow$

$$\lambda \underline{\omega} = \sum_i m_i r_i^2 \underline{\omega} - \sum m_i (\underline{r}_i \cdot \underline{\omega}) \underline{r}_i$$

$$\Rightarrow \left(\sum_i m_i r_i^2 - \lambda\right) \underline{\omega} = \sum_i m_i (\underline{r}_i \cdot \underline{\omega}) \underline{r}_i \longrightarrow (8)$$

If $\hat{a}$ is unit vector along axis corresponding to eigen value λ, then $\hat{a}$ is also a unit vector along $\underline{\omega}$ and we have

$$\underline{\omega} = \omega \hat{a}$$

using in (8)

$$\left(\sum_i m_i r_i^2 - \lambda\right) \omega \hat{a} = \sum_i m_i (\underline{r}_i \cdot \hat{a}) \underline{r}_i \omega$$

$$\left(\sum_i m_i r_i^2 - \lambda\right) \hat{a} = \sum_i m_i (\underline{r}_i \cdot \hat{a}) \underline{r}_i \longrightarrow (9)$$

Now if $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are unit vector along the principal axes corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively, then

$$\underline{r}_i = x_i\hat{a}_1 + y_i\hat{a}_2 + z_i\hat{a}_3$$

and from ⑨ we have for λ_1

$$\left(\sum_i m_i r_i^2 - \lambda_1\right)\hat{a}_1 = \sum_i m_i\left[(x_i\hat{a}_1 + y_i\hat{a}_2 + z_i\hat{a}_3)\cdot\hat{a}_1\right]\underline{r}_i$$

$$\left(\sum_i m_i r_i^2 - \lambda_1\right)\hat{a}_1 = \sum_i m_i x_i \underline{r}_i \longrightarrow (10)$$

Similarly for λ_2, λ_3

$$\left(\sum_i m_i r_i^2 - \lambda_2\right)\hat{a}_2 = \sum_i m_i y_i \underline{r}_i \longrightarrow (11)$$

$$\left(\sum_i m_i r_i^2 - \lambda_3\right)\hat{a}_3 = \sum_i m_i z_i \underline{r}_i \longrightarrow (12)$$

Comparing co-efficients of $\hat{a}_1, \hat{a}_2, \hat{a}_3$ in (10), (11), (12) we get

$$\sum_{i=1}^{n} m_i r_i^2 - \lambda_1 = \sum_i m_i x_i^2 \quad \& \quad \sum m_i x_i y_i = 0 = I_{12}$$

$$\sum_{i=1}^{n} m_i r_i^2 - \lambda_2 = \sum_i m_i y_i^2 \quad \sum m_i y_i z_i = 0 = I_{23}$$

$$\sum m_i x_i z_i = 0 = I_{13}$$

$$\sum_{i=1}^{n} m_i r_i^2 - \lambda_3 = \sum_i m_i z_i^2$$

$$\Rightarrow \lambda_1 = \sum_i m_i(y_i^2 + z_i^2) = I_{11} = I_1$$

$$\lambda_2 = \sum_i m_i(x_i^2 + z_i^2) = I_{22} = I_2$$

$$\lambda_3 = \sum_i m_i(x_i^2 + y_i^2) = I_{33} = I_3$$

Thus proved

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Method-II (Tensorial Method)

Theorem# Given the inertia tensor I_{ij} of a rigid body relative to Co-ordinate System $Ox_1x_2x_3$, where O is point about which body is rotating. The principal moments of inertia I_1', I_2', I_3' and principal direction of inertia $\underline{e}_1', \underline{e}_2', \underline{e}_3'$ (associated with principal co-ordinate system $Ox_1'x_2'x_3'$ at O) are given by the equations

$$\det(I_{ij} - I_k' \delta_{ij}) = 0$$

$$\text{and} \quad \Sigma (I_{ij} - I_k' \delta_{ij}) e_{k,j} = 0$$
$$k, i = 1, 2, 3$$

where $e_{k,j}$ are Components of the vector $\underline{e}_k$

Proof# Let $Ox_1'x_2'x_3'$ be principal Co-ordinate System at O and $\underline{e}_1', \underline{e}_2', \underline{e}_3'$ be unit vectors along these axes. Let I_{kl}' denote the Components of inertia tensor w.r.t Co-ordinate System $Ox_1'x_2'x_3'$ Then

$$I_{kl}' = a_{ki} a_{lj} I_{ij} \longrightarrow ①$$

But since the inertia tensor I_{kl}' relative to $Ox_1'x_2'x_3'$ is diagonal, therefor we can write

$$I_{kl}' = I_k' \delta_{kl} \longrightarrow ②$$

from ① & ②

$$a_{ki} a_{lj} I_{ij} = I_k' \delta_{kl} \longrightarrow ③$$

Multiplying both sides by a_{km} and summing over k

we have

$$(a_{km}\, a_{ki})\, a_{lj}\, I_{ij} = a_{km} I'_k\, \delta_{kl}$$

$$\delta_{mi}\, a_{lj}\, I_{ij} = I'_k\, \delta_{kl}\, a_{km}$$

$$\delta_{mi}\, a_{lj}\, I_{ij} = I'_l\, a_{lm} \qquad \because \delta_{kl} I'_k a_{km} = I'_l a_{lm}$$

$$\text{or} \quad a_{lj}\, I_{mj} = I'_l\, a_{lm}$$

$$= I'_l\, a_{lj}\, \delta_{jm} \qquad \because a_{lj}\delta_{jm} = a_{lm}$$

$$(I_{mj} - I'_l\, \delta_{jm})\, a_{lj} = 0 \longrightarrow (4)$$

Where j is a dummy index. For each value of l, we obtain from (4) a set a simultaneous linear equation for the direction cosines a_{1j}, a_{2j}, a_{3j} of the dashed axes. Solving these we find the directions of the axes of the system $ox'_1x'_2x'_3$ i.e principal directions. A necessary and sufficient condition that (4) have a non-trivial solution is

$$\det(I_{mj} - I'_l\, \delta_{mj}) = 0 \longrightarrow (5)$$

By solving the equations (5), we obtain the three values of principal moments of inertia I'_1, I'_2, I'_3. If we put one of these values of I' in (4), we will obtain the corresponding direction cosines a_{lj} of $ox'_1x'_2x'_3$

Remarks #(1) We have noted that finding eigen values of inertia matrix $[I]$ amounts to transformation to principal axes and eigen values I_1, I_2, I_3 are principal moment of inertia.

2) We have seen, that for any inertia tensor, the elements of which are computed for a given origin, it is possible to perform a rotation of the co-ordinate axes about that origin in such a way that the the inertia tensor becomes diagonal; the new co-ordinates axes are then the principal axes of the body and the new moments of inertia are the principal moments of inertia and are diagonal elements of diagonal inertia tensor. Thus for any body and for any choice of origin, there always exists a set of principal axes.

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Principal Axes Form an Orthogonal Set#

Theorem# By use of tensorial approach, prove that the principal axes form an orthogonal set.

Proof# Since the pricipal axes are found by solving secular equation of Inertia tensor, therefor let us assume that we have solved the secular equations and have determined the principal moments of inertia, all of which are distinct. Now, we know that for each principal moment there exists a corresponding principal axis which has the property that if angular velocity vector $\underline{\omega}$ lies along this axis, then angular momentum vector is similarly oriented. that is to each I_j, there corresponds an angular velocity $\underline{\omega}_j$, with components $\omega_{1j}, \omega_{2j}, \omega_{3j}$ (2nd subscript corresponds to principal moment concerned and 1st subscript gives components)

Thus, for mth principal moment, we have

$$L_{im} = I_m \omega_{im} \longrightarrow (1)$$

In terms of the elements of moment of inertia tensor, we have

$$L_{im} = \sum_{k=1}^{3} I_{ik} \omega_{km} \longrightarrow (2)$$

Combining (1) & (2), we have

$$\sum_{k=1}^{3} I_{ik} \omega_{km} = I_m \omega_{im} \longrightarrow (3)$$

Similarly for the nth principal moment, we have

$$\sum_{i=1}^{3} I_{ki} \omega_{in} = I_n \omega_{kn} \longrightarrow (4)$$

Multiplying (3) by ω_{in} and summing over i we have

$$\sum_{i,k} I_{ik} \omega_{km} \omega_{in} = \sum_{i} I_m \omega_{im} \omega_{in} \longrightarrow (5)$$

Multiplying (4) by ω_{km} and summing over k, we have

$$\sum_{i,k} I_{ki} \omega_{in} \omega_{km} = \sum_{k} I_n \omega_{kn} \omega_{km} \longrightarrow (6)$$

Since $I_{ik} = I_{ki}$ (symmetry of inertia tensor

Therefore L.H. sides of (5) & (6) are identical

Subtracting (6) from (5), we get

$$I_m \sum_{i} \omega_{im} \omega_{in} - I_n \sum_{k} {}_{km} \omega_{kn} = 0 \longrightarrow (7)$$

Since i, k are dummy, we can replace them

by l and obtain

$$(I_m - I_n) \sum_l \omega_{lm} \omega_{ln} = 0 \longrightarrow (8)$$

By hypothesis principal moments are distinct so that $I_m \neq I_n$ and we have from (8)

$$\sum_l \omega_{lm} \omega_{ln} = 0 \longrightarrow (9)$$

which is scalar product of vectors $\underline{\omega_m}, \underline{\omega_n}$

Hence

$$\underline{\omega_m} \cdot \underline{\omega_n} = 0$$

Since the principal moments I_m, I_n are arbitrary, we conclude that each pair of principal axes is perpendicular and three principal axes form an orthogonal set.

If there is a double root of secular equation so that principal moments are $I_1, I_2 = I_3$, then analysis above shows that the angular velocity vectors satisfy the relations

$$\underline{\omega_1} \perp \underline{\omega_2} \qquad \underline{\omega_1} \perp \underline{\omega_3}$$

but nothing can be said about the angle bet $\underline{\omega_2}$ and $\underline{\omega_3}$. However if $I_2 = I_3$, then body possess an axis of symmetry corresponding to I_1. Therefore $\underline{\omega_1}$ lies along symmetry axis and $\underline{\omega_2}, \underline{\omega_3}$ are required only to lie in the plane perpendicular to $\underline{\omega_1}$. Consequently, there is no loss of generality if we choose $\underline{\omega_2} \perp \underline{\omega_3}$. Thus the principal axes for a rigid body with an axis of symmetry can also ~~form~~ be chosen to be an orthogonal set.

Principal Moments of Inertia are Real

Theorem # By means of Tensors, prove that the principal moments of inertia are real and angular velocity vectors are also real

Proof # Principal moments of inertia are obtained by Secular equation of inertia tensor — a cubic equation. Mathematically, at least one of the roots of the cubic equation must be real because complex roots occur in Conjugate form i.e there may be two imaginary roots. But principal moments of inertia are eigen values of real symmetric tensor and are therefore real. Here we prove this result in another way. We assume the roots to be complex and use a procedure similar to above. But now we must also allow ω_{im} to be complex.

For mth moment of inertia, we have

$$\sum_k I_{ik}\,\omega_{km} = I_m\,\omega_{im} \longrightarrow (1)$$

$$\sum I_{ki}\,\omega_{in} = I_n\,\omega_{kn} \longrightarrow (2)$$

By taking Conjugate on both sides of (2)

$$\sum_i I^*_{ki}\,\omega^*_{in} = I^*_n\,\omega^*_{kn} \longrightarrow (3)$$

Multiplying 1st equation by ω^*_{in} and summing over i and multiplying the 2nd equation by ω_{km} and summing over k. The inertia tensor is real and its elements are real so that $I_{ik} = I^*_{ki}$ and we have

$$\sum_{k,i} I_{ik}\,\omega_{km}\,\omega^*_{in} = I_m\,\omega_{im}\,\omega^*_{in} \rightarrow (1)$$

$$\sum_{i,k} I^*_{ki}\, \omega^*_{in}\, \omega_{km} = I^*_m\, \omega^*_{kn}\, \omega_{km} \rightarrow (4)$$

Subtracting (4) from (3) and changing dummies, i, k to l, we have

$$(I_m - I^*_n) \sum_l \omega_{lm}\, \omega^*_{ln} = 0 \rightarrow (5)$$

For the case $m = n$

$$(I_m - I^*_m) \sum_l \omega_{lm}\, \omega^*_{lm} = 0 \rightarrow (6)$$

$$(I_m - I^*_m)\ \underline{\omega}_m \cdot \underline{\omega}^*_m = 0 \rightarrow (7)$$

But $\underline{\omega}_m \cdot \underline{\omega}^*_m = |\omega_m|^2 \geq 0$ By $z\bar{z} = |z|^2$

But in general $|\omega_m|^2 > 0$. Therefore (7) will be true if

$$I_m = I^*_m$$

$\Rightarrow$ principal moments of inertia are all real.

Since $\{I\}$ is real, the vectors ω_m must also be reall

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Remarks # In proofs above we have have made reference to the inertia tensor. The only properties of inertia tensor which are used are the facts the tensor is symmetric and elements are real. Therefore we may conclude that any real, symmetrical tensor, has the following properties

(a) Diagonalization may be accomplished by any appropriate rotation of axes i.e a similarity transform

(b) The eigen values are obtained as roots of the secular determinant and real

(c) The eigen vectors are real and orthogonal.

Role of Symmetry in Finding Principal Axes

For most of the problems in rigid body dynamics, the bodies are of some regular shape so that the principal axes can be determined merely by ~~merely~~ examining the symmetry axis of the body. e.g

(1) Any body which is solid of revolution (e.g a cylinderical rod) has one pricipal axis which lies along the symmetry (e.g the centre-line of the cylinderical rod) and the other two axes are in a plane perpendicular to the symmetry axis. clearly, since body is symmetric, the choice of angular placement of these two axis arbitrary. If the moment of inertia along the symmetry is I_1, then $I_2 = I_3$ for a solid of revolution i.e secular equation has a double root.

Top # A rigid body capable of rotation about an axis is generally called a top.

Symmetric or Symmetrical Top

If a rigid is capable of rotation about a symmetry axis, then it is called symmetrical top. OR if $I_1 = I_2 \neq I_3$, then body is symmetrical top

Asymmetrical Top

If the pricipal moments of inertia are call distinct i.e if $I_1 \neq I_2 \neq I_3$ i.e body is not symmetric about any axis, then it is called asymmetrical top.

Rotor # If a body has $I_1 = 0$, $I_2 = I_3$, it is called a rotor e.g, two point masses connected by a weightless shaft or a diatomic molecule.

Problem # Prove that if $\underline{\omega}$ points in some direction which is not to a principal axes, $\underline{L}$ possesses components which are perpendicular to that direction.

Sol # Let this arbitrary direction of $\underline{\omega}$ coincide with x-axis of body co-ordinate axes which are not principal axes. Then

$$\underline{\omega} = \omega_x \hat{i}$$

Now angular momentum $\underline{L}$ is given by

$$\underline{L} = \hat{i}(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) + \hat{j}(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) + \hat{k}(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)$$

$\because \quad \underline{\omega} = \omega_x \hat{i} \qquad \omega_y = 0 = \omega_z$

$$\underline{L} = \hat{i}(I_{xx}\omega_x) + \hat{j}(I_{yx}\omega_x) + \hat{k}(I_{zx}\omega_x)$$

Thus angular momentum $\underline{L}$ has components L_y, L_z which are perpendicular to x-axis i.e direction of $\underline{\omega}$

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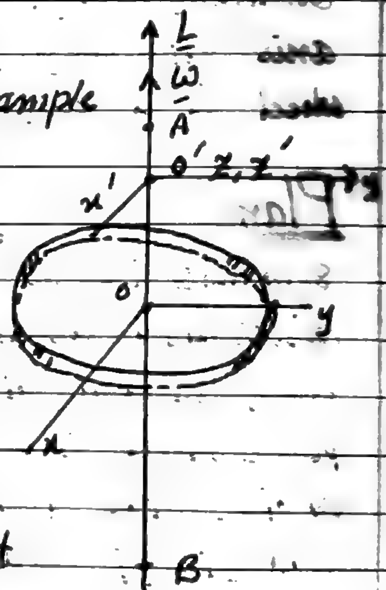
Important note # If a straight line is a principal axis at the centre of mass, it is a principal axis at all points on its length.

Usefulness of taking at least one of Co-ordinate Axes as the principlal axis at given origin

If we take at least one of the body Co-ordinate axes along a principal axis at a given point (origin), then our calculations become simple. Suppose at a given origin o of body system oz is a principal axis at o, then two products of inertia associated with z-axis, I_{xz}, I_{yz} would vanish & equation for angular momentum will become simple. Also if $\underline{\omega} = k\omega_z$, then whether or not ox, oy are pricipal axes $\underline{L}$ has a Component L_z along oz.

Explaination of the above Point#



Let us consider a simple example of a uniform wheel rotating about a fixed principal axis which is the axle through centre o about which rotation occurs. The axle AB is mounted in bearings at A & B and is attached to the centre, o of wheel. The axle taken as z-axis, is at least principal axes if the origin o is the centre of wheel because then z-axis is symmetry axis and also principal axis at any point of its length. For every element m_i at point (x_i, y_i, z_i) there exist another element m_j diagonally on opposite side of z-axis

~~(Such that)~~ at a point (x_j, y_j, z_j) Such that

$$x_j = -x_i \qquad y_j = -y_i \qquad z_j = z_i \quad \text{and } m_i = m_j$$

Consequently for such a symmetrical body
The quantities

$$I_{xz} = \sum_i^n m_i x_i z_i \quad \& \quad \sum_i^n m_i y_i z_i = I_{yz}$$

are both zero because particles are evenly and symmetrically distributed about centroidal axis and exactly half of n equal particles in $\sum_i^n m_i x_i z_i$ will have x-co-ordinates $-$ve of half other particles on opposite side of centroidal axis i.e

$m_1 = m$ $\quad$ $m_2 = m$ $\quad$ $\dots\ m_{\frac{n}{2}}$ $\quad$ $m_{\frac{n}{2}+1} = m$ $\quad$ $m_{\frac{n}{2}+2} = m$ $\quad \dots\ m_n = m$

(x_1, y_1, z_1) $\quad$ (x_2, y_2, z_1) $\quad \dots \quad$ $(-x_1, -y_1, z_1)$ $\quad$ $(-x_2, -y_2, z_2)$

and

$$\sum_i m_i x_i z_i = \sum_i m x_i z_i = (m x_1 z_1 + m x_2 z_2 + \dots - m x_1 z_1 - m x_2 z_2 - \dots\dots) = 0$$

Similarly $I_{yz} = 0$. Also z-axis is a principal axis at any point O' on its length because the wheel is a body of revolution about z-axis.

Plane Symmetry # Suppose a body has a plane symmetry i.e its particles are evenly and symmetrically distributed about a plane say xy-plane, then for every element m_i at (x_i, y_i, z_i) on one side of the plane, there is another element m_j at (x_j, y_j, z_j) located on other side of the plane such that $z_j = -z_i$ $x_i = x_j$ and $y_i = y_j$ and $m_i = m_j$. Accordingly for such a symmetrical body the quantities $I_{xz} = \sum_i m_i x_i z_i$ and $I_{yz} = \sum_i m_i y_i z_i$ are both zero. Thus the z-axis is principal axis at O. Further, it is important to notice that [illegible] perpendicular to this plane [illegible] point of intersection with

since in any such case I_{xz}, I_{yz} are both zero.

Summary #

1)# If a body has a plane symmetry, any axis perpendicular to this plane is a principal axis at the point of intersection with the plane.

2)# If a body is one of revolution about a given axis, the axis is a principal axis at all points its length.

3)# For a body which is a plane lamina, any axis which is perpendicular to the lamina is a principal axis at its intersection with the lamina. Clearly this is so because, on taking the z-axis as $\perp$ar to lamina, z-co-ordinate will be zero for all points of the lamina and $I_{xz} = 0 = I_{yz}$

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Moment of inertia about a Line when its Direction Cosines are given

Problem # Find the moment of inertia of a rigid body about a line (an axis) with D. Cosines λ, μ, ν when moments and product of inertia about some body co-ordinate axes are known

OR

Determine the moment of inertia of the distribution about the axis through O having D. Cosines λ, μ, ν in terms of D.Cs, moments of inertia and product of inertia relative to some co-ordinate axes at O

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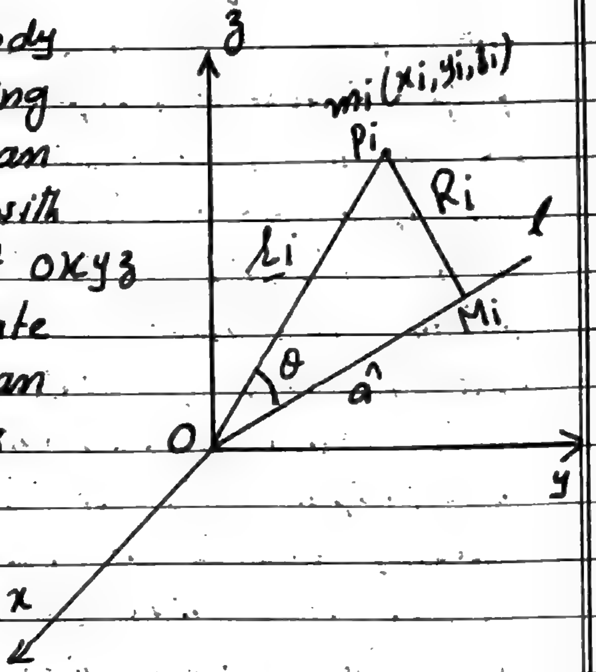
Also write K.E relative to this axis if point O is stationary point

Sol #

Consider a rigid body of n- particles rotating instantaneously about an axis through point O with angular velocity $\underline{\omega}$. Let $Oxyz$ be system of Co-ordinate axes at O and L is an axis through O with D.Cs λ, μ, ν.



Let $A = \sum_i m_i(y_i^2+z_i^2)$

$B = \sum_i m_i(x_i^2+z_i^2)$

$C = \sum_i m_i(x_i^2+y_i^2)$

$D = \sum m_i x_i y_i \qquad E = \sum m_i y_i z_i \qquad F = \sum m_i x_i z_i$

be moments and product of inertia relative to axes $Oxyz$.

Let $\hat{a}$ be unit vector along axis L, then

$$\hat{a} = [\lambda, \mu, \nu]$$

Let R_i be perpendicular distance of ith particle from L and $\underline{r}_i$ be P.V of particle from O.

$$R_i = |\underline{r}_i \times \hat{a}|$$

$$\underline{r}_i \times \hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ \lambda & \mu & \nu \end{vmatrix}$$

$$= i(\nu y_i - \mu z_i) + \hat{j}(\lambda z_i - \nu x_i) + \hat{k}(\mu x_i - \lambda y_i)$$

$$R_i^2 = [illegible] + (\lambda z_i - \nu x_i)^2 + (\mu x_i - \lambda y_i)^2$$

Moment of inertia about L is

$$I = \sum_i m_i R_i^2$$

$$= \sum_i m_i \left[(\nu y_i - \mu z_i)^2 + (\lambda z_i - \nu x_i)^2 + (\mu x_i - \lambda y_i)^2\right] \longrightarrow (1)$$

$$= \sum_i m_i \left[\nu^2(x_i^2+y_i^2) + \mu^2(x_i^2+z_i^2) + \lambda^2(y_i^2+z_i^2) - 2\mu\lambda x_i y_i - 2\nu\lambda x_i z_i - 2\mu\nu y_i z_i\right]$$

$$= \nu^2 \sum_i m_i (x_i^2+y_i^2) + \mu^2 \sum m_i (x_i^2+z_i^2) + \lambda^2 \sum_i m_i (y_i^2+z_i^2)$$
$$- 2\mu\lambda \sum_i m_i x_i y_i - 2\lambda\nu \sum x_i z_i m_i - 2\mu\nu \sum_i m_i y_i z_i$$

$$I = \nu^2 A + \mu^2 B + \lambda^2 C - 2\mu\lambda D - 2\lambda\nu E - 2\mu\nu F$$

K.E

Note that while calculating moment of inertia about L through O may be stationary or translating.

If O is stationary point, then K.E of the abody about L is calculated as

$$\underline{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$$

$$\underline{\omega} = \omega\hat{a} = \omega(\lambda\hat{i} + \mu\hat{j} + \nu\hat{k})$$

$$\underline{v}_i = \underline{\omega} \times \underline{r}_i$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ \lambda\omega & \mu\omega & \nu\omega \end{vmatrix}$$

$$= \omega\{(\nu y_i - \mu z_i)\hat{i} + \hat{j}(\lambda z_i - \nu x_i) + \hat{k}(\mu x_i - \lambda y_i)\}$$

$$|\underline{v}_i|^2 = |\underline{\omega} \times \underline{r}_i|^2$$

$$v_i^2 = \omega^2[(\nu y_i - \mu z_i)^2 + (\lambda z_i - \nu x_i)^2 + (\mu x_i - \lambda y_i)^2]$$

$$T = \frac{1}{2}\sum_i m_i v_i^2$$

$$= \frac{1}{2}\sum_i m_i \omega^2[(\nu y_i - \mu z_i)^2 + (\lambda z_i - \nu x_i)^2 + (\mu x_i - \lambda y_i)^2]$$

$$= \frac{1}{2}\omega^2 \sum m_i[(\nu y_i - \mu z_i)^2 + (\lambda z_i - \nu x_i)^2 + (\mu x_i - \lambda y_i)^2]$$

$$= \frac{1}{2}\omega^2 I \quad \text{, where from ①}$$

$$I = \sum m_i[(\nu y_i - \mu z_i)^2 + (\lambda z_i - \nu x_i)^2 + (\mu x_i - \lambda y_i)^2]$$

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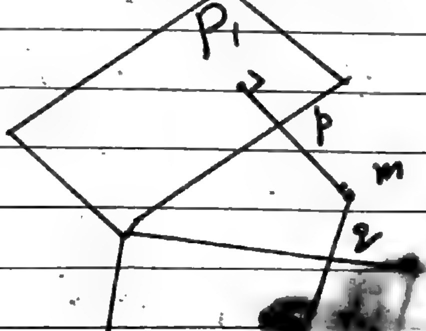
Product of Inertia of Particle about a Pair of Planes#

The product of inertia of a particle about a paire of planes P_1, P_2 at distance p, q from planes is

$$I_{pq} = -mpq$$



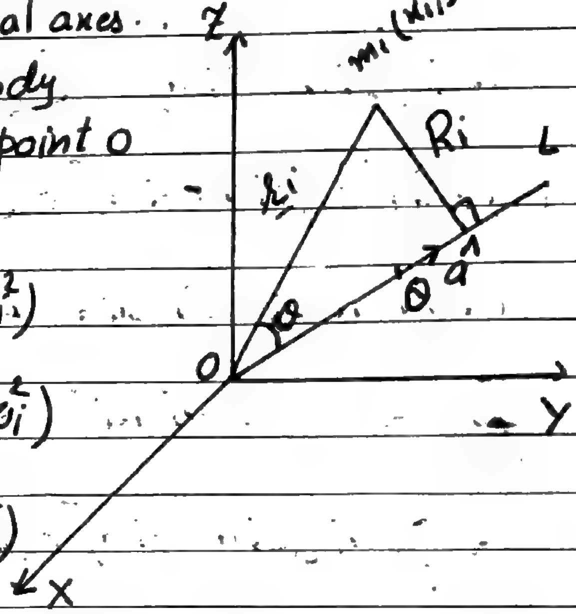
Momental Ellipsoid#

Problem #(a)* Derive the equation of momental ellipsoid for a rigid & Check the effect of rotation of axes on it.

(b)# Also derive equation of momental ellipsoid in tensorial form and prove that it is independent of the choice of the Co-ordinate system

(c)*# How will you prove with the help of momental ellipsoid that at every point of a body there are always at least three principal axes.

(d)*# Deduce momental ellipsoid for Plane distribution of Mass and identify it. Also deduce formula for the D. of principal axes.



Sol# Let OXYZ be body Co-ordinate system at point O of body

Let

$$I_{xx} = A = \sum_i m_i(y_i^2 + z_i^2)$$

$$I_{yy} = B = \sum_i m_i(z_i^2 + x_i^2)$$

$$I_{zz} = C = \sum_i m_i(x_i^2 + y_i^2)$$

Similarly the products of inertia about pairs of Co-ordinate plane (XY, XZ), (XY, YZ), (XZ, YZ) be

$$I_{xz} = -\sum_i m_i x_i z_i = G$$

x_i is distance of m_i from YZ plane & z_i is from XY plane

$$I_{yz} = -\sum_i m_i y_i z_i = F$$

$$I_{xy} = -\sum_i m_i x_i y_i = H$$

find moment of inertia of body about =

line L with D-cosines l, m, n and $\hat{a} = [l, m, n]$ unit vector along it.

Let $\underline{r}_i$ be p.v of mass particle m_i w.r.t O and R_i be its distance from L. Then

$$R_i = r_i \sin\theta = |\underline{r}_i \times \hat{a}|$$

$$R_i^2 = |\underline{r}_i \times \hat{a}|^2$$

$$\underline{r}_i \times \hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ l & m & n \end{vmatrix}$$

$$= i(y_i n - z_i m) + \hat{j}(l z_i - n x_i) + \hat{k}(x_i m - y_i l)$$

$$|\underline{r}_i \times \hat{a}|^2 = (y_i n + z_i m)^2 + (l z_i - n x_i)^2 + (x_i m - y_i l)^2$$

Moment of inertia I about L is

$$I = \sum m_i R_i^2 = \sum_i m_i |\underline{r}_i \times \hat{a}|^2$$

$$= \sum_i m_i \left[(y_i n - z_i m)^2 + (l z_i - n x_i)^2 + (x_i m - y_i l)^2\right]$$

$$= \sum_i m_i \left[n^2(x_i^2 + y_i^2) + m^2(x_i^2 + z_i^2) + l^2(y_i^2 + z_i^2) - 2ln x_i z_i - 2mn y_i z_i - 2lm x_i y_i\right]$$

$$= \sum_i m_i (x_i^2 + y_i^2) n^2 + \sum_i m_i (x_i^2 + z_i^2) m^2 + \sum_i m_i (y_i^2 + z_i^2) l^2 - 2ln \sum x_i z_i m_i - 2mn \sum_i y_i z_i m_i - 2lm \sum_i m_i x_i y_i$$

$$= l^2 A + m^2 B + n^2 C + lnG + 2mnF + 2lmH.$$

$I = Al^2 + Bm^2 + Cn^2 + 2lnG + 2mnF + 2lmH \rightarrow (1)$
which express the moment of inertia about line L in terms of the moments and product of inertia about the co-ordinate axes.

In fig let $Q(x, y, z)$ be a point on L and $\overrightarrow{OQ} = r$ which moves about O in any manner and let its length be variable so that for any instantaneous orientation of $\overrightarrow{OQ}$ (or line L) the moment of inertia about OQ is inversely proportional to r^2 i.e

$$I_{OQ} \propto \frac{1}{OQ^2}$$

$$(2) \leftarrow I_{OQ} = \frac{k}{r^2} \quad \text{where } k \text{ is constant}$$

$$\Rightarrow r^2 = \frac{k}{I_{OQ}} \Rightarrow r = \sqrt{\frac{k}{I_{OQ}}}$$

(otherwise we cannot obtain a surface of 2nd degree

Also $l = x/r \quad m = y/r, \ n = z/r \rightarrow (3)$
using (2) & (3) in (1)

$$k/r^2 = \frac{1}{r^2}\left[Ax^2 + By^2 + Cz^2 + 2xzG + 2yzF + 2xyH\right]$$

$$\Rightarrow Ax^2 + By^2 + Cz^2 + 2xzG + 2yzF + 2xyH = k \rightarrow (4)$$

which is a quadratic surface about.

Since for a fite rigid body, there is no orientation of L (OQ) for which $I_{OQ} = 0$ and $r = \infty$, therefore the surface must define an ellipsoid. K is an arbitrary [illegible]ence (4) represents a family of ellipsoids

, one for each value of k. The surface of ellipsoid is generated by the tip of vector r.
Usually it is convenient to take value of k equal to 1.

Equation of ellipsoid then becomes

$$Ax^2 + By^2 + Cz^2 + 2xzG + 2yzF + 2xyH = 1 \quad \rightarrow (5)$$

This ellipsoid is called Poinsot's ellipsoid of inertia of the body at O. It is fixed in body with centre at the origin O.

It should be noted that for $k = 1$, the moment of inertia I about OQ is

$$I_{OQ} = \frac{1}{r^2} \Rightarrow r = \frac{1}{\sqrt{I_{OQ}}}$$

If the axes Ox, Oy, Oz relative to which momental ellipsoid is found are principal axes of the body, then

$G = F = H = 0$ and equation of ellipsoid becomes

$$Ax^2 + By^2 + Cz^2 = 1 \quad \rightarrow (6)$$

which is equation w.r.t principal diameters of ellipsoid.

from $r = \frac{1}{\sqrt{I_{OQ}}}$

let $r_1 = \frac{1}{\sqrt{A}}$, $r_2 = \frac{1}{\sqrt{B}}$, $r_3 = \frac{1}{\sqrt{C}}$

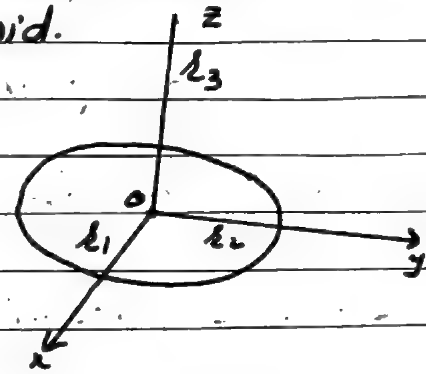
$\Rightarrow A = \frac{1}{r_1^2}$ $\quad B = \frac{1}{r_2^2}$ $\quad C = \frac{1}{r_3^2}$

Equation (6) becomes

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} + \frac{z^2}{r_3^2} = 1$$

where r_1, r_2, r_3 are three fixed [illegible]

values of r, which gives the lengths of semi-minor axis, the intermediate axis and semi-major axes of the ellipsoid.



From $r = \dfrac{1}{\sqrt{I_{oo}}}$

It is clear that the moment of inertia is maximum about the axis for which r is min and vice versa

Effect of Rotation of Axes on Momental Ellipsoid

When the axes at o are rotated to new axes $OX'Y'Z'$, then definitely A, B, C, G, F, H will also be changed. Let their new values be A', B', C' G', F', H'. Then the equation of transformed ellipsoid is

$$A'x'^2 + B'y'^2 + C'z'^2 + 2G'x'z' + 2F'y'z' + 2H'x'y' = 1$$

which is again an ellipsoid. Thus with rotation of axes equation of ellipsoid remain an equation of ellipsoid i.e is independent of the choice of Co-ordinate system.

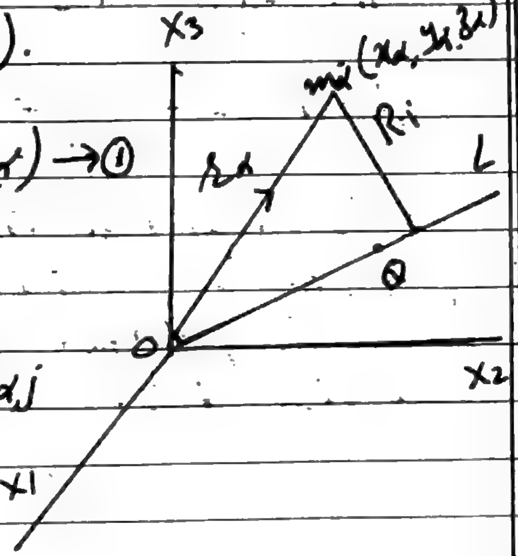
(b) Equation of Ellipsoid in Tensorial Form

For easy notation in tensorial form we take axes $OXYZ$ to $OX_1Y_2Z_3$ and D. Cosines l, m, n of line L to be $n_1 = l$ $n_2 = m$ $n_3 = n$. Then $\hat{a} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$. The moment of inertia of the body about L is given by

$$I = \sum_\alpha m_\alpha \, |\hat{a} \times \underline{r}_\alpha|^2$$

$$= \sum_\alpha m_\alpha (\hat{a} \times \underline{r}_\alpha).$$

$$= \sum_\alpha m_\alpha (\hat{a} \times \underline{r}_\alpha) \cdot (\hat{a} \times \underline{r}_\alpha) \rightarrow ①$$



In tensor notation

$(\cancel{\hat{a} \times \underline{r}_\alpha}) \cdot (\hat{a} \times \underline{r}_\alpha) = \epsilon_{ijk} \, n_i \, r_{\alpha,j}$

where $r_{\alpha,j} = x_{\alpha,j}$ are components of $\underline{r}_\alpha$

Also dot product $\underline{A} \cdot \underline{A}$ in tensor form is

$$\underline{A} \cdot \underline{A} = \sum_k A_k A_k \quad (k \text{ is dummy})$$

using this we have

$$(\hat{a} \times \underline{r}_\alpha) \cdot (\hat{a} \times \underline{r}_\alpha) = \sum_k (\hat{a} \times \underline{r}_\alpha)_k \, (\hat{a} \times \underline{r}_\alpha)_k$$

$$= (\hat{a} \times \underline{r}_\alpha)_k \, (\hat{a} \times \underline{r}_\alpha)_k$$

$$= \epsilon_{ijk} \, n_i \, x_{\alpha,j} \; \epsilon_{lmk} \, n_l \, x_{\alpha,m}$$

$$= \epsilon_{ijk} \, \epsilon_{lmk} \, n_i \, x_{\alpha,j} \; n_l \, x_{\alpha,m}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \, n_i x_{\alpha,j} \; n_l \, x_{\alpha,m} \rightarrow ②$$

$\epsilon_{ijk} \, \epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

using ② in ①

$$I = \sum_\alpha m_\alpha (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \, n_i x_{\alpha,j} \, n_l \, x_{\alpha,m}$$

$$= \sum_\alpha m_\alpha \, \delta_{il}\delta_{jm} \, n_i \, x_{\alpha,j} \, n_l \, x_{\alpha,m} - \sum_\alpha m_\alpha \, \delta_{im}\delta_{jl} \, n_i \, x \ldots$$

$$= \sum_{\alpha} m_\alpha n_i n_i x_{\alpha,j} x_{\alpha,j} - \sum_{\alpha} m_\alpha n_i n_j x_{\alpha,i} x_{\alpha,j}$$

using $n_i = \sum_j \delta_{ij} n_j = \delta_{ij} n_j$ & $x_{\alpha,j} x_{\alpha,j} = \sum_{\alpha} x_j^2 = r_\alpha^2$

$$I = \sum_{\alpha} m_\alpha n_i n_j \delta_{ij} (r_\alpha^2) - \sum_{\alpha} m_\alpha n_i n_j x_{\alpha,i} x_{\alpha,j}$$

$$= \sum_{\alpha} m_\alpha n_i n_j \left[r_\alpha^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j} \right]$$

$$= \sum_{ij} n_i n_j I_{ij} \longrightarrow (2)$$

where $I_{ij} = \sum_{\alpha} \left[r_\alpha^2 \delta_{ij} - x_{\alpha,i} x_{\alpha,j} \right] m_\alpha$

$$I_L = \sum_{i,j} n_i n_j I_{ij} \longrightarrow (3)$$

Now let Q be a point on L such that

$\underline{r} = \overrightarrow{OQ}$ and its magnitude is such that

$$I_{OQ} \propto \frac{1}{r^2}$$

$$I_{OQ} = \frac{k}{r^2} \longrightarrow (4)$$

$$\underline{r} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}$$

and $n_1 = \dfrac{x_1}{r}$ $\quad n_2 = \dfrac{x_2}{r}$ $\quad n_3 = \dfrac{x_3}{r}$

$$\Rightarrow n_i = \frac{x_i}{r} \qquad n_j = \frac{x_j}{r} \longrightarrow (5)$$

from (3) moment of inertia about OQ is

$$I_{OQ} = \sum_{i,j} n_i n_j I_{ij}$$

using (4) & (5)

$$= \frac{1}{r^2} \sum_{i,j} x_i x_j I_{ij}$$

$$\sum_{i,j} x_i x_j I_{ij} = k \longrightarrow (6)$$

which is the required inertia ellipsoid at O in tensorial form

Since tensor equation retains its form in every Co-ordinate system obtained by rotation about O, therefore equation (6) will be true in every Co-ordinate system i.e its form will be independent of the choice of Co-ordinate system. Hence for a given k, there is a unique ellipsoid for a given inertia tensor.

From the knowledge of inertia tensor we can determine the inertia ellipsoid and vice versa. The inertia ellipsoid w.r.t a system of Co-ordinates $OX_1X_2X_3$ can be used to calculate the moment of inertia about any axis through O.

Poinsot's ellipsoid of inertia of body at O is given by

$$\sum_{i,j} x_i x_j I_{ij} = 1 \longrightarrow (7)$$

(C) Principal Axes and Momental Ellipsoid

If the principal (symmetry) diameters of the inertia ellipsoid are taken as co-ordinates axes, the the products of inertia relative to these axes are zero due to symmetry about these axes. Now every ellipsoid has at least Three principal diameters and it is always possible to rot (or to orient) the axes until they coincide wi these diameters and then become prin

of the body at point at which momental ellipsoid is taken. Hence at every point of body there are always at least three principal axes. Proved.

(d) Deduction of Momental Ellipsoid For Plane Distribution of Mass

Momental ellipsoid at o is given by

$$Ax^2+By^2+Cz^2+2xzG+2yzF+2xyH=k \longrightarrow ①$$

In case of plane distribution of mass (plane lamina), there will no body z-axis and we have zero z-co-ordinate for each mass particle. so

$$G = \sum_i m_i x_i z_i = \sum_i m_i x_i (0) = 0$$

$$F = \sum_i m_i y_i z_i = \sum_i m_i y_i (0) = 0$$

and equation of momental ellipsoid becomes

$$Ax^2+By^2+Cz^2+2xyH = k = \text{Constant}.$$

$$Ax^2+By^2+C(0)^2+2xyH = \text{Constant}$$

$$Ax^2+By^2+2xyH = \text{Constant} \longrightarrow ②$$

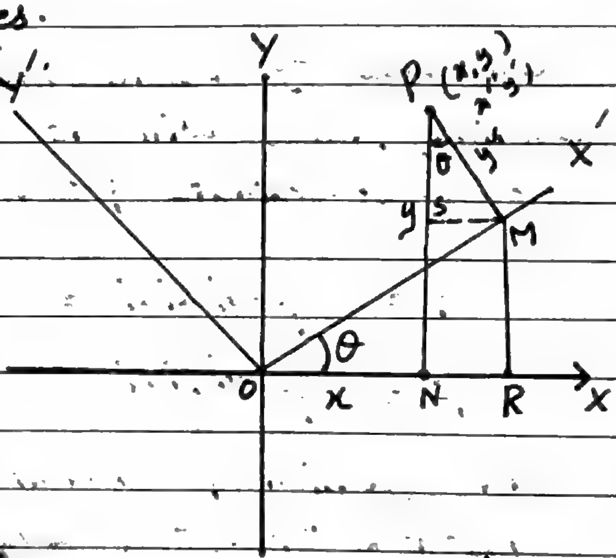
which is equation of ellipse and is called momental ellipse.

If equation ② is transformed such that its major and minor axes (which are symmetry axes) are co-ordinate axes at o, we get standard equation of ellipse whose major and minor axes become principal axes of the body at o.

Now to transform ② to standard form (i.e. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$) we are to rotate the Co-ordinate axes through such an angle for which the product term xy in new Co-ordinate system is eliminated (ie is zero). This process is equivalent to finding principal axes and such value of angle will give the direction of principal axes relative to old axes.

Suppose θ is the angle through which axes are rotated so that product term becomes zero. Let $OX'Y'$ be new axes.

If (x,y) & (x',y') are Co-ordinates of point relative co-ordinate systems OXY, $OX'Y'$, then



$x = ON = x'\cos\theta - y'\sin\theta$

$y = PN = x'\sin\theta + y'\cos\theta$

Using these in ②

$OM = x'$,
$PM = y'$
$ON = x$
$NR = SM = y'\sin\theta$
$OR = OM\cos\theta = x'\cos\theta$
$MR = OM\sin\theta = x'\sin\theta = SN$

$$A(x'\cos\theta - y'\sin\theta)^2 + B(x'\sin\theta + y'\cos\theta)^2 + 2(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)H = \text{Constant}$$

$$Ax'^2 + By'^2 + (-A+B)x'y'\cos\theta\sin\theta + 2H[x'^2\sin\theta\cos\theta - y'^2\sin\theta\cos\theta + x'y'\cos^2\theta - x'y'\sin^2\theta) = \text{Constant}$$

$$Ax'^2 + By'^2 + [(-A+B)\sin 2\theta + 2H\cos 2\theta]x'y' + 2H[x'^2\sin\theta\cos\theta - y'^2\sin\theta\cos\theta] = \text{Constant} \rightarrow ③$$

Now $x'y'$ term will be eliminated if its co-efficient is zero i.e

$$(-A+B)\sin 2\theta + 2H\cos 2\theta = 0$$

$$-(A-B)\sin 2\theta + 2H\cos 2\theta = 0$$

$$(A-B)\sin 2\theta = 2H\cos 2\theta$$

$$\tan 2\theta = \frac{2H}{A-B}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2H}{A-B}\right) \longrightarrow ④$$

where $H = -\sum_i m_i x_i y_i$

under this condition equation ③ becomes

$$(A+2H\sin\theta\cos\theta)x_1'^2 + (B-2H\sin\theta\cos\theta)y'^2 = \text{Constant}$$

$$A'x'^2 + B'y'^2 = \text{Constant}$$

which can be reduced to standard form
The new axes are principal axes and their direction relative to old axes is given by ④

<u>Note</u> # In order to tranform equation of momental ellipsoid ① to standard form. (i.e $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$) we are to eliminate the product terms in x, y, z. Now ① can be written as

$$[x, y, z]\begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k$$

when product terms are eliminated, we get

$$[x' \; y', z']\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = k$$

[illegible]k to diagonalise the inertia matrix [illegible]

Remarks # It is useful to consider the space of vector $\underline{r} = \overrightarrow{OQ} = x\hat{i} + y\hat{j} + z\hat{k}$ or space of the co-cordinates x, y, z in terms of which inertia ellipsoid is described & is super-imposed upon the ordinary space co-ordinates $OXYZ$. They have common origin and have Ox ‖al to OX, Oy parallel to OY and Oz parallel to OZ

Some Special Cases of Momental Ellipsoid

Case I # If all the particles of the system lie on a given line, then momental ellipsoid is cylinder with given line as its axis.

Proof # Let the z-axis is given line on which all particles lie and any point it be origin

$$A = \sum_i m_i(y_i^2 + z_i^2)$$

$$= \sum_i m_i z_i^2 \qquad \because y_i = x_i = 0$$

$$B = \sum_i m_i(x_i^2 + z_i^2)$$

$$= \sum_i m_i z_i^2 \qquad \because x_i = 0$$

$$C = \sum_i m_i(x_i^2 + y_i^2) = 0 \qquad \because x_i = y_i = 0$$

$$G = F = H = 0 \qquad \because x_i = y_i = 0$$

We note that $A = B$

Equation of momental ellipsoid reduces to

$$A(x^2 + y^2) = k$$

$$x^2 + y^2 = \text{Constant}$$

which is a cylinder and z-axis is its

Case-II # (Degenerate Case)

Momental ellipsoid when Co-ordinate axes are principal axes is given by

$$Ax^2 + By^2 + Cz^2 = k$$

where A, B, C are principal moments

If all principal moments are equal, then

$$A = B = C$$

and equation of ellipsoid is

$$Ax^2 + Ay^2 + Az^2 = k$$

$$x^2 + y^2 + z^2 = \frac{k}{A}$$

$$(x-0)^2 + (y-0)^2 + (z-0)^2 = \left(\sqrt{\frac{k}{A}}\right)^2$$

It is an ellipsoid in which all semi-axes are equal and it degenerate in a sphere with centre at $O(0,0,0)$ and radius $\sqrt{\frac{k}{A}}$.

In this case all axes passing through centre O are principal axes and all the moments are equal. Thus

Result# When the inertia ellipsoid w.r.t a point O of body is a sphere, all axes passing through O are principal axes and have identical moments of inertia which are equal to the reciprocal of the square of the radius r of the inertial sphere

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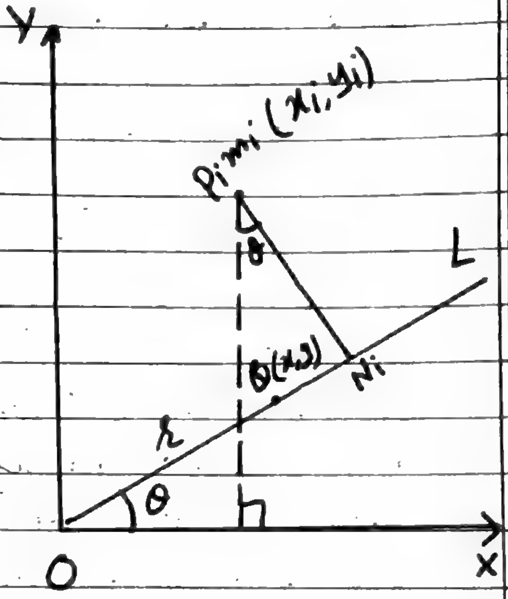
Plane Distribution of Mass

Problem # Derive an expression of moment of inertia about a line inclined at some angle with the x-axis at any point of plane lamina. Hence equation of momental ellipse. How will you determine the direction of principal axes at a point and principal moments of inertia relative to principal axes #

Sol # Let OX, OY be perpendicular axes at point O of plane distribution of mass (plane lamina).



Let $A = \sum_i m_i y_i^2$

$B = \sum_i m_i x_i^2$

$F = \sum m_i x_i y_i$

Let m_i be mass particle at point $P_i(x_i, y_i)$ and P_iN_i is its $\perp$ar distance of m_i from a line L inclined at angle θ to x-axis

slope of line $= m = \tan\theta = \frac{\sin\theta}{\cos\theta}$

It passes through $(0,0)$.

Its equation is

$$y - 0 = \frac{\sin\theta}{\cos\theta}(x - 0)$$

$$y\cos\theta - x\sin\theta = 0$$

PN_i = Distance of point P_i from line

$$P_iN_i = \frac{|y_i\cos\theta - x_i\sin\theta|}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$$|P_iN_i|^2 = (y_i\cos\theta - x_i\sin\theta)^2$$

Moment of inertia of distribution about line L is

$$I = \sum_i m_i|P_iN_i|^2$$

$$= \sum_i m_i(y_i\cos\theta - x_i\sin\theta)^2$$

$$= \sum_i m_i[y_i^2\cos^2\theta + x_i^2\sin^2\theta - 2x_iy_i\sin\theta\cos\theta]$$

$$= \left(\sum_i m_iy_i^2\right)\cos^2\theta + \left(\sum_i m_ix_i^2\right)\sin^2\theta - 2\left(\sum_i m_ix_iy_i\right)\sin\theta\cos\theta$$

$$= A\cos^2\theta + B\sin^2\theta - 2F\sin\theta\cos\theta \longrightarrow ①$$

Now let a point Q on line L such that

$$OQ = r$$

and $I_{OQ} \propto \frac{1}{r^2}$

$$\Rightarrow I_{OQ} = \frac{k}{r^2} \longrightarrow ②$$

From ①

$$I_{OQ} = A\cos^2\theta + B\sin^2\theta - 2F\sin\theta\cos\theta$$

using ②

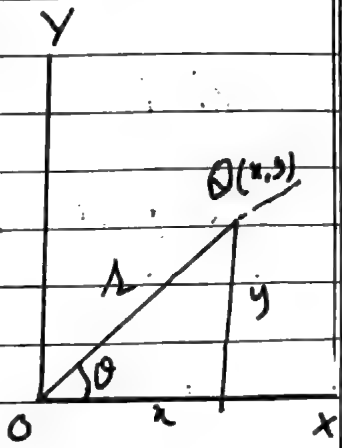
$$A\cos^2\theta + B\sin^2\theta - 2F\sin\theta\cos\theta$$

$$K = r^2\left(A\cos^2\theta + B\sin^2\theta - 2F\sin\theta\cos\theta\right)$$

if (x, y) are Co-ordinates of Q, then

$\cos\theta = x/r \qquad \sin\theta = y/r$

using these



$$K = r^2\left(A\frac{x^2}{r^2} + \frac{By^2}{r^2} - \frac{2Fxy}{r^2}\right)$$

$$\Rightarrow Ax^2 + By^2 - 2Fxy = k \longrightarrow (3)$$

$\because$ A & B are both +ve i-e have same signs

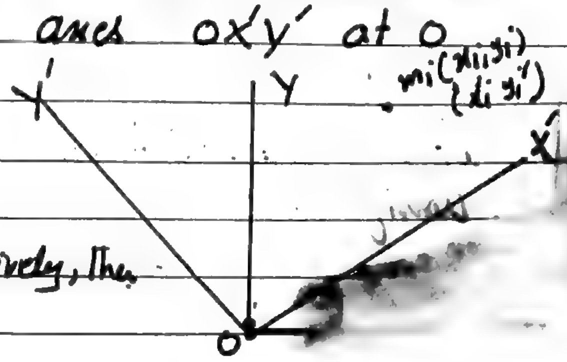
$\therefore$ (3) is an equation of ellipse and is called momental ellipse at O

When Co-ordinates axes are taken as principal axes of this ellipse, then product xy is eliminated i-e $F=0$ and the axes become principal axes of the body at O. Now can always rotate Co-ordinates axes till they coincide with the principal axes of the ellipse and become principal axes.

Direction of Principal axes at O

Suppose the axes OXY are rotated through angle α about O and the rotated axes Coincide with principal axes $OX'Y'$ at O

If (x_i, y_i) & (x_i', y_i') are Co-ordinates of ith particle w.r.t axes OXY, $OX'Y'$ respectively, then



$x_i = x_i' \cos\alpha - y_i' \sin\alpha \quad x_i' = y_i \cos\alpha + y_i \sin\alpha$

$y_i = x_i' \sin\alpha + y_i' \cos\alpha$

$\cancel{I_{x'y'} = \sum_{i=1} m_i x_i' y_i'}$

$\cancel{= \sum_i m_i (x_i' \cos\alpha - y_i' \sin\alpha)(x_i' \sin\alpha + y_i' \cos\alpha)}$

$x_i' = x_i \cos\alpha + y_i \sin\alpha$

$y_i' = y_i \cos\alpha - x_i \sin\alpha$

$$I_{x'y'} = \sum_i m_i x_i' y_i'$$

$$= \sum_i m_i (x_i \cos\alpha + y_i \sin\alpha)(y_i \cos\alpha - x_i \sin\alpha)$$

$$= \sum_i m_i [\, x_i y_i \cos^2\alpha - x_i^2 \sin\alpha \cos\alpha + y_i^2 \sin\alpha \cos\alpha - x_i y_i \sin^2\alpha \,]$$

$$= \sum_i m_i (y_i^2 - x_i^2) \sin\alpha \cos\alpha + \sum m_i x_i y_i (\cos^2\alpha - \sin^2\alpha)$$

$$= (A - B) \sin\alpha \cos\alpha + F \cos 2\alpha$$

$$= \frac{1}{2}(A - B) \sin 2\alpha + F \cos 2\alpha$$

$\because$ $OX'Y'$ are principal axes

so $I_{x'y'} = 0$

$\ldots B) \sin 2\alpha + F \cos 2\alpha = 0$

$$\frac{1}{2}(A-B)\sin 2\alpha = -F\cos 2\alpha$$

$$\tan 2\alpha = -\frac{2F}{A-B}$$

$$\alpha = \frac{1}{2}\tan^{-1}\left(-\frac{2F}{A-B}\right) \rightarrow (4)$$

If we take $F = -\Sigma m_i x_i y_i$, then

$$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2F}{A-B}\right) \rightarrow (5)$$

But $\tan 2\alpha = \tan(2\alpha+\pi) = \tan 2(\alpha+\frac{\pi}{2})$

$\Rightarrow \alpha+\frac{\pi}{2}$ is also a direction of principal axis which is OY'

If $B > A$, then (4) shows that $2\alpha, \alpha$ are acute

Now with the help of (1)

$$I_{ox'} = A\cos^2\alpha - 2F\sin\alpha\cos\alpha + B\sin^2\alpha \rightarrow (6)$$

$$I_{oy'} = A\cos^2(\tfrac{\pi}{2}+\alpha) - 2F\sin(\tfrac{\pi}{2}+\alpha)\cos(\tfrac{\pi}{2}+\alpha) + B\sin^2(\tfrac{\pi}{2}+\alpha)$$

$$= A\sin^2\alpha + 2F\sin\alpha\cos\alpha + B\cos^2\alpha \rightarrow (7)$$

from (6)

$$I_{ox'} = A\left(\frac{1+\cos 2\alpha}{2}\right) - F\sin 2\alpha + B\left(\frac{1-\cos 2\alpha}{2}\right)$$

$$I_{ox'} = \frac{1}{2}(A+B) - \left[\frac{1}{2}(B-A)\cos 2\alpha + F\sin 2\alpha\right]$$

$$= \frac{1}{2}(A+B) - \frac{1}{2}\left[(B-A)\cos 2\alpha + 2F\sin 2\alpha\right] \rightarrow (8)$$

from (7)

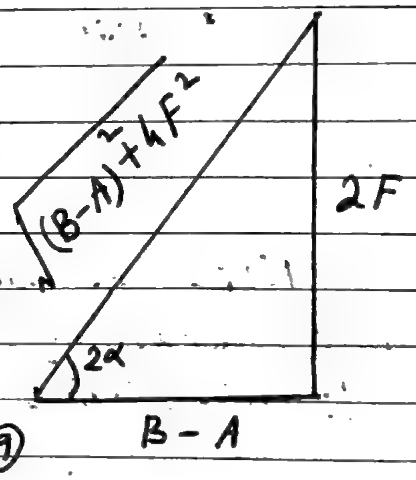
$$I_{oy'} = A\left(\frac{1-\cos 2\alpha}{2}\right) + F\sin 2\alpha + B\left(\frac{1+\cos 2\alpha}{2}\right)$$

$$= \frac{1}{2}(A+B) + \frac{1}{2}\left[(B-A)\cos 2\alpha + 2F\sin 2\alpha\right] \rightarrow (9)$$

But $\tan 2\alpha = \dfrac{2F}{B-A}$

$$\sin 2\alpha = \frac{2F}{\sqrt{(B-A)^2 + 4F^2}}$$

$$\cos 2\alpha = \frac{B-A}{\sqrt{(B-A)^2 + 4F^2}}$$



Using these values in (8) & (9)

$$A' = I_{ox'} = \frac{1}{2}(A+B) - \frac{1}{2}\left[\frac{(B-A)(B-A)}{\sqrt{(B-A)^2+4F^2}} + \frac{4F^2}{\sqrt{(B-A)^2+4F^2}}\right]$$

$$= \frac{1}{2}(A+B) - \frac{1}{2}\left[\frac{(B-A)^2 + 4F^2}{\sqrt{(B-A)^2+4F^2}}\right]$$

$$= \frac{1}{2}(A+B) - \frac{1}{2}\sqrt{(B-A)^2 + 4F^2} \rightarrow (10)$$

$$I_{oy'} = \frac{1}{2}(A+B) + \frac{1}{2}\sqrt{(B-A)^2+4F^2} = B' \rightarrow (11)$$

we note from (10) & (11)

$B' > A'$

$\Rightarrow$ A' is min. & B' is maximum.

$\Rightarrow$ The greatest and the least moments of inertia for lines through O are attained along the principal axes. Also

$$C' = A' + B'$$

Problems About K.E, Angular Velocity and Angular Momentum

Problem # What is the K.E of a homogeneous solid circular cylinder of mass m and radius a, rolling upon a plane with linear velocity V

Sol # Note Moment of inertia of a solid homogeneous cylinder about its axis

$= \frac{1}{2}$ Mass (radius)^2

Momement of inertia of a hollow uniform circular cylinder about its axis = Mass (radius)^2

Here we may consider rotation of solid cylinder about its axis which passes through its C.G. C and velocity V of cylinder can be taken also velocity of C.G.

Relative to a fixed point O, K.E is given by

$$T_o = T_c + \frac{1}{2} m V^2$$

where V is velocity of c.m. relative to O and T_c is rotational K.E. relative to c.m.

$$T_O = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$$

But $\quad V = a\omega \qquad I = \frac{1}{2}ma^2$

$$T_O = \frac{1}{2}mv^2 + \frac{1}{2}\cdot\frac{1}{2}ma^2\left(\frac{V}{a}\right)^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2$$

In case of hollow uniform cylinder we have

$$I = ma^2$$

$$T_O = \frac{1}{2}mv^2 + \frac{1}{2}ma^2\left(\frac{V}{a}\right)^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$$

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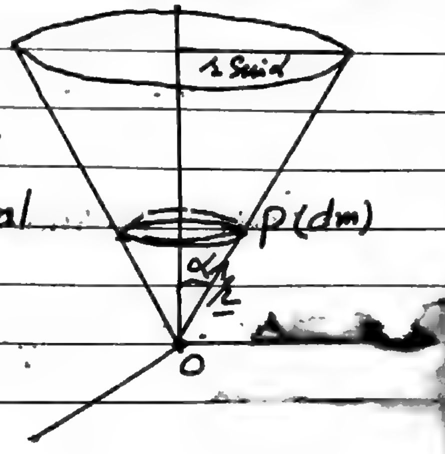
Problem#* A rod of length $2a$ and mass m turns about one end O describing a cone with semi-vertical angle α. It completes one revolution in time T. Find the direction and magnitude of the angular momentum about O



Sol# Note# Rotation about a fixed axis called precession

Let $\dot{\varphi}$ be the precessional speed about fixed point O.

Then

$$\dot{\varphi} = \frac{2\pi}{T} = \omega$$

The angular momentum of an infinitesimal mass element dm of rod about O is given by

$$d\underline{L} = \underline{r} \times \underline{v}\, dm$$

where $\underline{v}$ is the direction of increase of φ and its magnitude is given by

$$v = |\underline{v}| = r \sin\alpha\, \dot{\varphi} \qquad v = r\omega$$

The angular momentum of dm is $\perp$ar to both $\underline{r}$ & $\underline{v}$ and therefore it is in the vertical plane containing $\underline{r}$. The resultant angular momentum will therefore be in the same vertical plane.

The magnitude of the angular momentum is given by

$$|d\underline{L}| = |\underline{r} \times \underline{v}|\, dm$$

$$= r v \sin 90^\circ\, dm \qquad \underline{r} \perp \underline{v}$$

$$= r v\, dm$$

putting $v = r \sin\alpha\, \dot{\varphi} = r \sin\alpha \cdot \frac{2\pi}{T}$

$$dL = \frac{2\pi}{T} \sin\alpha\, r^2\, dm$$

But $\rho = \frac{dm}{dr} \Rightarrow dm = \rho\, dr$

$$dL = \frac{2\pi}{T} \sin\alpha \cdot r^2 \rho\, dr$$

Total angular momentum is given by -

$$L = \frac{2\pi}{T} \sin\alpha \cdot \rho \int_0^{2a} r^2\, dr$$

$$= \frac{2\pi \sin\alpha}{T} \rho \left| \frac{r^3}{3} \right|_0^{2a}$$

$$= \frac{2\pi \sin\alpha}{T} \rho \cdot \left[\frac{(2a)^3}{3} - 0 \right]$$

$$= \frac{2\pi \sin\alpha \rho}{3T} \cdot 8a^3 = \left(\frac{16\pi a^3 \sin\alpha}{3T} \right) \rho$$

$$\rho = \frac{\text{Mass}}{\text{length}} = \frac{m}{2a}$$

$$L = \left(\frac{16\pi a^3 \sin\alpha}{3T} \right) \times \frac{m}{2a}$$

$$L = \frac{8\pi a^2 \sin\alpha}{3T}$$

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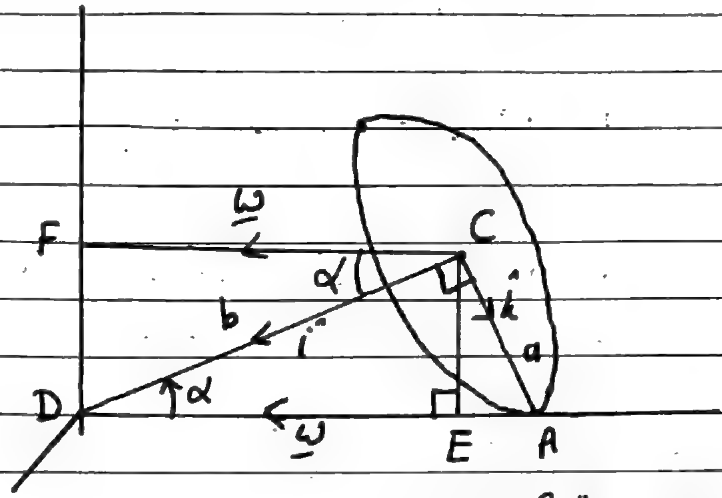
Problem* A uniform circular disc of radius "a" and mass "m" is rigidly mounted on one end of a shaft CD of length "b", the shaft is normal to the disk at the centre C. The disc rolls on a rough horizontal plane D being fixed in the plane by a smooth joint. If the centre of the disc rotates about the vertical through D with constant angular speed n. Find the angular velocity, the K.E and angular momentum of disk about D.

Sol* Moment of inertia of uniform circular disc about centroidal axis perpendicular to the plane of disc $= \frac{1}{2}(\text{Mass})(\text{radius})^2$

M.I about any diameter $= I_d = \frac{1}{4}(\text{Mass})(\text{radius})^2$



CD = b $\qquad$ CA = a = radius of the disc

From right-angled triangle ACD

$$DA = \sqrt{a^2+b^2}$$

$$\Rightarrow \cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$$

The instantaneous linear velocity of C is $\perp$ar to the plane ADC and its magnitude along CF & DE is

$$V = n \cdot CE = n \cdot DF$$

$$= n\, b\cos\alpha$$

$$= \frac{nb \cdot b}{\sqrt{a^2+b^2}} \qquad \because \cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$$

Instantaneously the disc rotates about DA, C describing a circle of radius EC. Therefore instantaneous angular velocity ω about DA is

$$\omega = \frac{v}{CE} = \frac{nb\cos\alpha}{b\sin\alpha}$$

$$= n\cot\alpha$$

Let $\hat{i}, \hat{k}$ be unit vectors parallel to CD & CA ($\hat{j}$ will be unit vector $\perp$ar to ACD)

Then

Compont of $\underline{\omega}$ along CD $= \omega \cos\alpha \, \hat{i}$

Compont of $\underline{\omega}$ along CA $= \omega \cos(90+\alpha)\hat{k}$

$$= -\omega \sin\alpha \, \hat{k}$$

$$\underline{\omega} = \omega \cos\alpha \, \hat{i} - \omega \sin\alpha \, \hat{k}$$

$$= \omega \left(\cos\alpha \, \hat{i} - \sin\alpha \, \hat{k} \right)$$

$$= \frac{nb}{a} \left(\cos\alpha \, \hat{i} - \sin\alpha \, \hat{k} \right)$$

$$= \frac{nb}{a} \left[\frac{b}{\sqrt{a^2+b^2}} \hat{i} - \frac{a}{\sqrt{a^2+b^2}} \hat{k} \right]$$

$$= \frac{nb^2}{a\sqrt{a^2+b^2}} \hat{i} - \frac{nb}{\sqrt{a^2+b^2}} \hat{k}$$

$$= \omega_x \hat{i} + \omega_z \hat{k}$$

K.E relative to D is given by

$$T_D = T_c + \frac{1}{2} m v^2$$

$$= \frac{1}{2} m v^2 + T_c$$

$$\frac{1}{2} m v^2 + T_c$$

Now CA, CD and an axis perpendicular to these are principal axes of the disc

M.I of disc about CD $= I_1 = \frac{1}{2}(\text{mass})(\text{radius})^2$

$$I_1 = \frac{1}{2} m a^2$$

M.I of disc about AC $= I_d = I_3 = \frac{1}{4} m a^2$

Also M.I about an axis $\perp$ar to ACD $= I_2 = 0$

So Rotational K.E relative to principal axes is

$$T_c = \frac{1}{2}(I_1 \omega_x + I_2 \omega_y + I_3 \omega_z)$$

$$= \frac{1}{2}(I_1 \omega_x^2 + 0 + I_3 \omega_z^2)$$

$$= \frac{1}{2}\left[\frac{1}{2} m a^2 \left(\frac{n b^2}{a\sqrt{a^2+b^2}}\right)^2 + \frac{m a^2}{4}\left(\frac{-nb}{\sqrt{a^2+b^2}}\right)^2\right.$$

$$= \frac{m a^2}{4} \cdot \frac{n^2 b^4}{a^2(a^2+b^2)} + \frac{m a^2}{8} \frac{n^2 b^2}{a^2+b^2}$$

$$= \frac{m n^2 b^4}{4(a^2+b^2)} + \frac{m n^2 b^2 a^2}{8(a^2+b^2)}$$

$$= \frac{a^2 m n^2 b^2}{8(a^2+b^2)} + \frac{m^2 n^2 b^4}{4(a^2+b^2)}$$

$$T_D = \frac{1}{2} m v^2 + T_c$$

$$= \frac{1}{2} m \left(\frac{n b^2}{\sqrt{a^2+b^2}}\right)^2 + \frac{a^2 m n^2 b^2}{8(a^2+b^2)} + \frac{m^2 n^2 b^4}{4(a^2+b^2)}$$

$$= \frac{m n^2 b^4}{2(a^2+b^2)} + \frac{a^2}{8} \frac{m n^2 b^2}{(a^2+b^2)}$$

$$= \frac{mn^2b^2}{8(a^2+b^2)}\left[4b^2+a^2+2b^2\right]$$

$$= \frac{mn^2b^2}{8(a^2+b^2)}\left[a^2+6b^2\right]$$

The angular momentum about D is

$$\underline{L}_D = \underline{L}_C + \vec{DC} \times m\underline{v}$$

$$= \underline{L}_C + (-b\hat{i}) \times m\underline{v}$$

$$\underline{v} = \frac{nb^2}{\sqrt{a^2+b^2}}\hat{j}$$

$$\underline{L}_D = (I_1\omega_x\hat{i} + I_2\omega_y\hat{j} + I_3\omega_z\hat{k}) - b\hat{i} \times \frac{mnb^2}{\sqrt{a^2+b^2}}\hat{j}$$

$$= I_1\omega_x\hat{i} + 0 + I_3\omega_z\hat{k} - \frac{mnb^3}{\sqrt{a^2+b^2}}\hat{k}$$

$$= \frac{1}{2}ma^2\left(\frac{nb^2}{a\sqrt{a^2+b^2}}\right)\hat{i} + \frac{1}{4}ma^2\left(\frac{-nb}{\sqrt{a^2+b^2}}\right)\hat{k} - \frac{mnb^3}{\sqrt{a^2+b^2}}\hat{k}$$

$$= -\frac{mnb^3}{\sqrt{a^2+b^2}}\hat{k} - \frac{mna^2b}{4\sqrt{a^2+b^2}}\hat{k} + \frac{mnab^2}{\sqrt{a^2+b^2}}\hat{i}$$

$$= \frac{mnab^2}{\sqrt{a^2+b^2}}\hat{i} - \frac{mnb}{4\sqrt{a^2+b^2}}\left(a^2+4b^2\right)\hat{k}$$

By Muhammad Hussain Lecturer (Maths)

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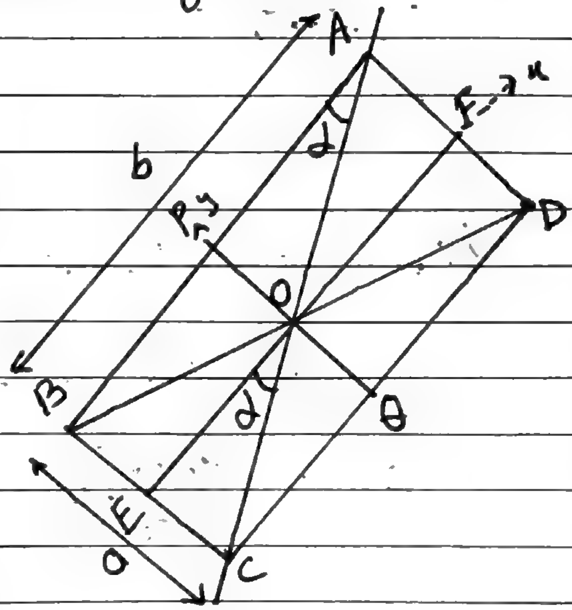
Problem # A thin rectangular sheet of length b and width (a) is rotating about one of its diagonals with a uniform angular speed ω. Find the direction and magnitude of the angular momentum



Sol # Referred to axes through centre and parallel to the edges.

$A = I_{EF} = I_{xx}$

$= \frac{1}{3}(\text{Mass})(\text{half of width})^2$

$= \frac{1}{3} M \left(a/2\right)^2 = \frac{Ma^2}{12}$

$B = I_{PQ} = I_{yy} = \frac{Mb^2}{12}$

$F = I_{xy} = 0$ because axes are symmetry axes or principal axes

Moment of inertia about diagonal AC is given by

$$I_{AC} = A\cos^2\alpha + B\sin^2\alpha + F\sin 2\alpha$$

$$= A\cos^2\alpha + B\sin^2\alpha + 0 \qquad \because F = 0$$

from ΔABC $\qquad AC = \sqrt{a^2+b^2}$

$\cos\alpha = \frac{b}{\sqrt{a^2+b^2}} \qquad \sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$

$$I_{AC} = \frac{Ma^2}{12}\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 + \frac{Mb^2}{12}\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2$$

$$= \frac{Ma^2b^2}{12(a^2+b^2)} + \frac{Ma^2b^2}{12(a^2+b^2)}$$

$$= \frac{Ma^2b^2}{6(a^2+b^2)}$$

Therefore the magnitude of angular momentum about fixed axis AC is

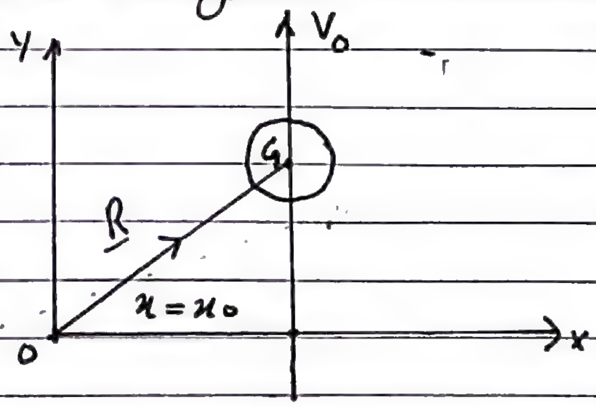
$$L = I\dot{\theta} = I\omega$$

$$= \frac{Ma^2b^2}{6(a^2+b^2)}\omega$$

The direction of angular momentum is clearly along fixed axis

---

Problem # A thin uniform circular disc of radius (a) and mass m lies in the xy-plane and rotates with a constant angular velocity ω about an axis through the centre and parallel to z-axis. Simultaneously the centre moves with uniform linear velocity (speed) V_0 parallel to y-axis at distance x_0 from y-axis. Find the angular momentum about origin



Sol # M.I of the disc about centroidal axis and $\perp$ar to the plane of disc is given by

$$M.I = \frac{1}{2}(\text{Mass})(\text{radius})^2$$

$$I = \frac{1}{2}ma^2$$

P.V of c.m $= \underline{R} = [x_0, y, 0]$

Velocity $= \underline{V_0} = [0, V_0, 0]$

The angular momentum about O is given by

$$\underline{L}_o = \underline{L}_c + m\underline{R} \times \underline{v}_o$$

$$= I\omega\hat{k} + m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_0 & y & 0 \\ 0 & V_0 & 0 \end{vmatrix}$$

$$= I\omega\hat{k} + m[x_0 V_0 \hat{k}] \qquad \underline{L}_c = I\omega\hat{k}$$

$$= \left(\frac{1}{2}ma^2\omega + mx_0V_0\right)\hat{k}$$

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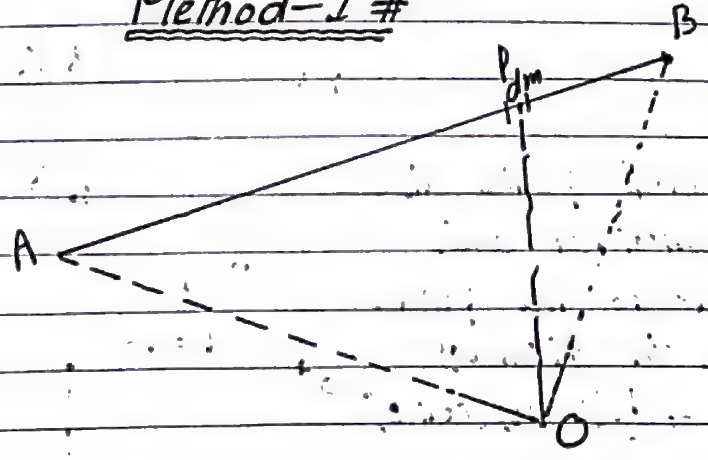
Problem*# A uniform rod AB of mass M moves so that A and B have velocities u_A, u_B at any instant. Show that the K.E T of the rod is given by

$$T = \frac{M}{6}\left[u_A^2 + u_A \cdot u_B + u_B^2\right]$$

Sol# Method-I#



Consider an infinitesimal mass element dm at point P on AB and

$$\vec{AP} = \lambda \vec{AB}$$

Then $0 \leq \lambda \leq 1$

$$\vec{AP} = \lambda(\vec{AP} + \vec{PB})$$

$$(1-\lambda)\vec{AP} = \lambda \vec{PB}$$

$$(1-\lambda)[\vec{OP} - \vec{OA}] = \lambda[\vec{OB} - \vec{OP}]$$

$$\Rightarrow (1-\lambda)\vec{OA} + \lambda \vec{OB} = \vec{OP}$$

Diff w.r.t t

$$(1-\lambda)\frac{d\vec{OA}}{dt} + \lambda \frac{d\vec{OB}}{dt} = \frac{d}{dt}(\vec{OP})$$

$$(1-\lambda)\underline{u}_A + \lambda \underline{u}_B = \underline{v}_P$$

$$|\underline{v}_P|^2 = \underline{v}_P \cdot \underline{v}_P$$

$$= [(1-\lambda)\underline{u}_A + \lambda \underline{u}_B] \cdot [(1-\lambda)\underline{u}_A + \lambda \underline{u}_B]$$

$$v_P^2 = (1-\lambda)^2 u_A \cdot u_A + \lambda^2 \underline{u}_B \cdot \underline{u}_B + 2\lambda(1-\lambda)(\underline{u}_A \cdot \underline{u}_B)$$

Let m be the mass of length AP. Then

$$m = \lambda M$$

$$\Rightarrow dm = d\lambda\, M$$

Now K.E of the mass dm is given by

$$dT = \frac{1}{2} v_P^2\, dm$$

$$dT = \frac{1}{2} v_p^2 M d\lambda$$

$$= \frac{1}{2}\left[(1-\lambda)^2 u_A^2 + \lambda^2 u_B^2 + 2\lambda(1-\lambda)(\underline{u}_A \cdot \underline{u}_B)\right] M d\lambda$$

Total K.E of the rod is

$$T = \frac{1}{2}\int_0^1 \left[(1-\lambda)^2 u_A^2 + \lambda^2 u_B^2 + 2\lambda(1-\lambda)(\underline{u}_A \cdot \underline{u}_B)\right] M\, d\lambda$$

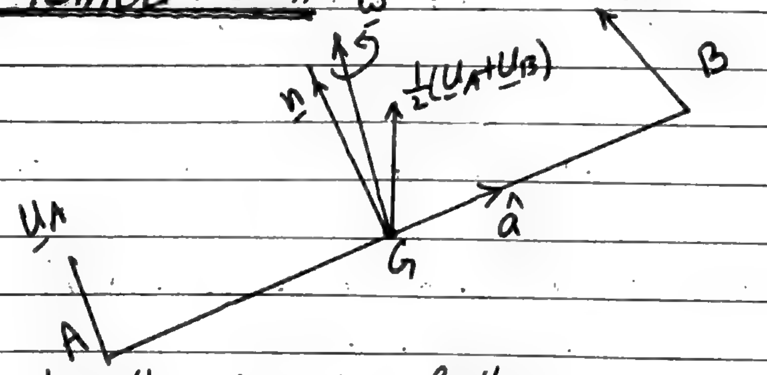
$$= \frac{1}{2}M\int_0^1 (1-\lambda)^2 u_A^2\, d\lambda + \frac{M}{2}\int_0^1 \lambda^2 u_B^2\, d\lambda + M(\underline{u}_A \cdot \underline{u}_B)\int_0^1 (\lambda-\lambda^2)\, d\lambda$$

$$= -\frac{1}{2}M u_A^2\left[\frac{(1-\lambda)^3}{3}\right]_0^1 + \frac{M}{6}\left[\lambda^3\right]_0^1 u_B^2 + M(\underline{u}_A \cdot \underline{u}_B)\left[\frac{\lambda^2}{2} - \frac{\lambda^3}{3}\right]_0^1$$

$$= \frac{1}{6}M u_A^2 + \frac{M}{6}u_B^2 + \frac{1}{6}M\, \underline{u}_A \cdot \underline{u}_B$$

$$= \frac{M}{6}\left[u_A^2 + u_B^2 + \underline{u}_A \cdot \underline{u}_B\right]$$

Method- II



Let $2a$ be the length of the rod and $\hat{a}$ unit vector along $\overrightarrow{AB}$. Let $\underline{\omega}$ be angular velocity of the rod and $\underline{n}$ unit normal to the rod through C.G, G and in the plane of $\underline{\omega}$, $\hat{a}$

Velocity of $G = \frac{1}{2}(\underline{u}_A + \underline{u}_B) = \underline{v}$ (say)

$\because$ Whole of the mass of rod is distributed along it

$\therefore$ AB is a principal axis, $\underline{n}$ the 2nd principal axis and third principal axis will be $\hat{a} \times \underline{n}$

If M is the mass of rod, then principal moments of inertia about these three axes are

$I_{AB} = A = 0 = I_{xx}$ Symmetry axis

$I_n = B = \frac{1}{3}Ma^2 = I_{yy}$

About 3rd axis $= C = \frac{1}{3}Ma^2 = I_{zz}$

Angular velocity Components are

$\omega_x =$ along AB $= \underline{\omega} \cdot \hat{a}$

$\omega_y =$ along $\underline{n}$ $= \underline{\omega} \cdot \hat{n}$

$\omega_z =$ along axis which is $\perp$ar to plane of $\hat{a}$ & $\hat{n}$

$= \omega \cos 90 = 0$

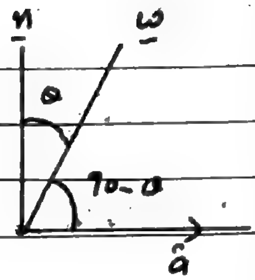
K.E of the rod is

$$T = T_c + \frac{1}{2}[I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2]$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}[(0)(\underline{\omega}\cdot\hat{a})^2 + \frac{1}{3}Ma^2(\underline{\omega}\cdot\hat{n})^2 + \frac{1}{3}Ma^2(0)]$$

$$= \frac{1}{2}MV^2 + \frac{1}{6}Ma^2|\underline{\omega}\cdot\underline{n}|^2$$

$$= \frac{1}{2}MV^2 + \frac{1}{6}Ma^2|\underline{\omega}\cdot\underline{n}|^2 \longrightarrow (1)$$



Now $\underline{\omega}\cdot\underline{n} = \omega\cos\theta$

$= \omega\sin(90^\circ-\theta)$

$= |\underline{\omega}\times\hat{a}|$

$\Rightarrow |\underline{\omega}\cdot n|^2 = |\underline{\omega}\times\hat{a}|^2$

But $a^2|\underline{\omega}\cdot n|^2 = a^2|\underline{\omega}\times\hat{a}|^2$

$= |a\underline{\omega}\times\hat{a}|^2$

$= |\underline{\omega}\times a\hat{a}|^2$

$= |\underline{\omega}\times\underline{a}|^2$

$= |\underline{\omega}\times \underline{GB}|^2 = |\text{Velocity of B relative to G}|^2$

$= |\underline{U}_B - \frac{1}{2}(\underline{U}_A+\underline{U}_B)|^2$

$= |\frac{1}{2}(\underline{U}_B-\underline{U}_A)|^2$

$= \frac{1}{4}(\underline{U}_B-\underline{U}_A)\cdot(\underline{U}_B-\underline{U}_A)$

$= \frac{1}{4}[U_B^2 - 2\underline{U}_A\cdot\underline{U}_B + U_A^2]$

using in ①

$T = \frac{1}{2}MV^2 + \frac{1}{24}[U_B^2 - 2\underline{U}_A\cdot\underline{U}_B + U_A^2]M$

$= \frac{1}{2}M\{\frac{1}{2}(\underline{U}_A+\underline{U}_B)\}^2 + \frac{1}{24}[U_B^2 - 2\underline{U}_A\cdot\underline{U}_B + U_A^2]M$

$= \frac{1}{8}M[U_A^2 + U_B^2 + 2\underline{U}_A\cdot\underline{U}_B] + \frac{1}{24}M[U_A^2 + U_B^2 - 2\underline{U}_A\cdot\underline{U}_B]$

$= \frac{1}{6}MU_A^2 + \frac{1}{6}M\,U_A\cdot U_B + \frac{1}{6}MU_B^2$

$$= \frac{1}{6} M \left(u_A^2 + u_B^2 + \underline{u}_A \cdot \underline{u}_B \right)$$

Problem # Find the K.E of a circular disc of radius 2 ~~meter~~ ft and weight 10 lbs, rolling, without sliding, on a horizontal plane such that its centre moves forward with a speed of 20 ft/sec.

Sol # K.E = K.E due to translation + K.E due to rotation.

Here $m = 10\,lbs$ radius $= r = 2\,ft$

$v =$ velocity of c.m $= 20\,ft/sec$

$$\omega = \frac{v}{r} = \frac{20}{2} = 10 \text{ radians/sec}$$

K.E due to translation $= \frac{1}{2} m v^2$

$$= \frac{1}{2} \times 10 \times (20)^2 = 2000 \text{ poundals}$$

K.E due to rotation $= \frac{1}{2} I \omega^2$

where $I =$ M.I of disc about centroidal axis ⊥ar to its plane $= \frac{1}{2} m r^2 = \frac{1}{2}(10)(4) = 20$

$$T_{rot} = \frac{1}{2}(20)(10)^2 = 1000 \text{ poundals}$$

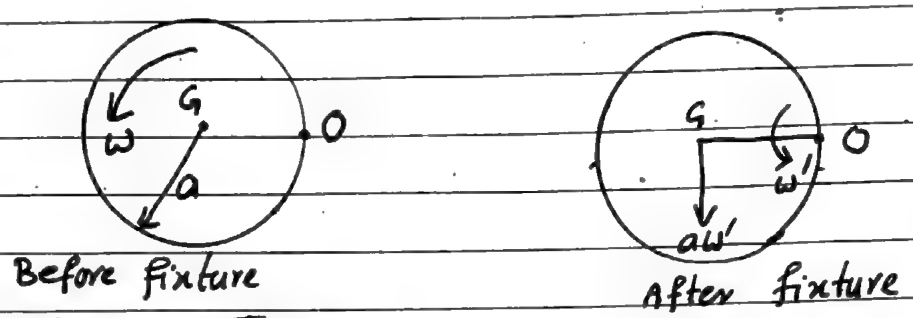
Hence total K.E is given by

$$= \frac{2000 + 1000}{32} = \frac{375}{4} = 93\tfrac{3}{4} \text{ ft-lb}$$

Problem # A uniform circular disc of mass M and radius 'a' is rotating in its plane with initial angular velocity ω, its centre being at rest. If a point on its rim be suddenly fixed, find the new angular velocity of the disc and

and the velocity of its centre.

Sol#



Before FixTure#

Suppose O is the point on the rim of disc, which will be fixed.

The angular momentum befor fixture about c.m G = $0 + \frac{1}{2} I\omega$

where I = m.I of disk about a centroidal axis perpendicular to the plane of disc

$= \frac{1}{2} ma^2$

Angular momentum $= \frac{1}{2} ma^2\omega$

After Fixture

Suppose now point O on the rim is suddenly fixed and the disc has angular velocity ω' about O.

The velocity of centroid relative to O $= a\omega' = u$

The angular momentum about point O is now given by

$$L_O = L_G + \cancel{I_G}\omega' + mua = I\omega' + mua$$

where I_G = M.I of discl about an axis Through G and perpendicular to the plane of disc

$= I_G = \frac{1}{2} ma^2$

$$L_O = \frac{1}{2}ma^2\omega' + ma^2\omega' \quad \because u = a\omega'$$

$$= \frac{3}{2}ma^2\omega'$$

OR

We can find the angular momentum of the disc about O directly as

$$L_O = I_O\,\omega'$$

Here I_O = M.I of disc about an axis through O and $\perp$ar to the plane of disc

$$= I_G + ma^2 = \frac{1}{2}ma^2 + ma^2 = \frac{3}{2}ma^2$$

$$L_O = \frac{3}{2}ma^2\omega'$$

Now the impulse that causes the change must act through O which becomes fixed, therefore there can be no change of angular momentum about O. We have

total momentum befor finture = total momentum after finture

$$\frac{3}{2}ma^2\omega' = \frac{1}{2}ma^2\omega$$

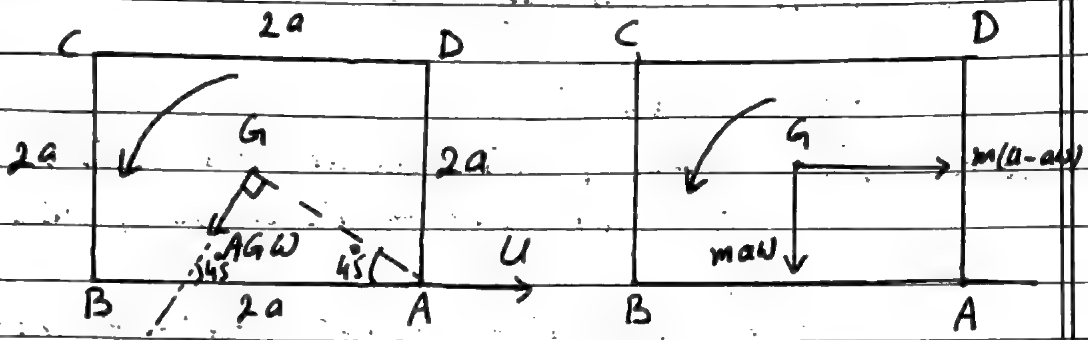
$$\omega' = \frac{1}{3}\omega$$

Hence, immediately after fixing, the disc rotates with angular velocity $\frac{1}{3}\omega$ about O and velocity of centroid $= u = a\omega' = \frac{1}{3}a\omega$

By Muhammad Hussain Lecturer (Maths) Govt College Asghar Mall RWP # 0300581 4930

Problem# A square lamina ABCD rests on a smooth horizontal plane. If the corner A is made to move with velocity u along the line BA produced, determine the initial angular velocity of lamina.

Sol#



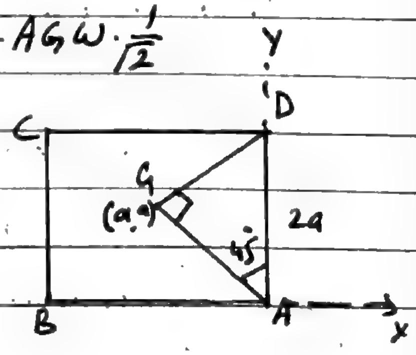
Let m be the mass and $2a$ a side of square lamina. If ω is the angular velocity of lamina, then velocity of C.G., G relative to point A is

$$v = AG\omega.$$

The compont of v along BA is

$$u - v\cos 45^\circ = u - AG\omega \cdot \frac{1}{\sqrt{2}}$$

$$AG = 2a\cos 45^\circ$$
$$= 2a \cdot \frac{1}{\sqrt{2}}$$



Compont of v along BA is

$$= u - \frac{2a}{\sqrt{2}}\omega \cdot \frac{1}{\sqrt{2}}$$

$$= u - a\omega$$

The component of v along AD $= AG\omega\cos 45^\circ$

$$= \frac{2a}{\sqrt{2}}\omega \cdot \frac{1}{\sqrt{2}} = a\omega$$

Now taking A as origin, co-ordinates of G

are (a, a)

$$\overrightarrow{AG} = -a\hat{i} + a\hat{j}$$

Velocity of G is

$$\underline{v} = (U - a\omega)\hat{i} - a\omega\hat{j}$$

$U - a\omega$ along BA & $a\omega$ along DA

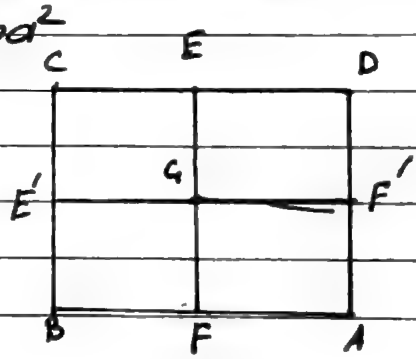
Angular momentum about A is

$$\underline{L}_A = \underline{L}_G + \overrightarrow{AG} \times m\underline{v}$$

$$\underline{L}_G = I_G \omega \hat{k}$$

where I_G = M.I of Lamina about an axis through G and $\perp$ar to the plane of lamina

$$= \frac{1}{3}ma^2 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$



Note here $I_{EF} = I_{E'F'} = \frac{1}{3}ma^2$ and by $\perp$ar axes Theorem we have

$$I_G = \frac{2}{3}ma^2$$

$$\underline{L}_G = \frac{2}{3}ma^2\omega\hat{k}$$

Also

$$\overrightarrow{AG} \times m\underline{v} = m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & a & 0 \\ U - a\omega & -a\omega & 0 \end{vmatrix}$$

$$= m\left[a^2\omega - a(U - a\omega)\right]\hat{k}$$

$$= \left[ma^2\omega - ma(U - a\omega)\right]\hat{k}$$

So,

$$\underline{L}_A = \left[\frac{2}{3}ma^2\omega + ma^2\omega - ma(U - a\omega)\right]\hat{k}$$

But only the external impulsive action is applied at A and there is no moment force at A, therefore the momentum about A is zero

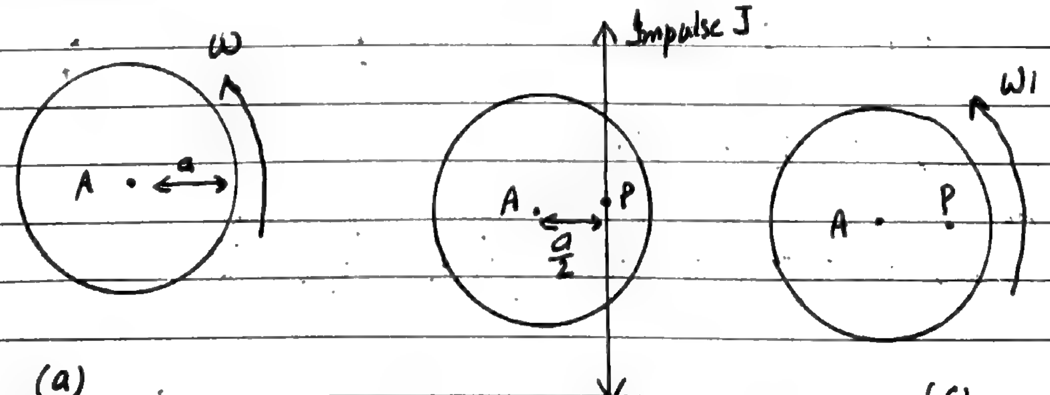
$$\Rightarrow \quad \underline{h}_A = 0$$

$$\Rightarrow \tfrac{2}{3} m a^2 \omega - m a (u - a\omega) + m a^2 \omega = 0 \quad \because \hat{k} \neq 0$$

$$\Rightarrow \quad \omega = \frac{3u}{8a}$$

Problem # A uniform circular disc of mass m and radius (a) is rotating with a constant angular velocity ω in a horizontal plane about a vertical axis through its centre A. A particle P of mass 2m is placed gently on a disc at a distance a/2 from A. If the particle does not slip on the disc, find the new angular velocity of the rotating system.

Sol #



(a) Rotation of disc alone

(b) Instantaneous addition of particle

(c) Rotation of disc & particle

The M.I of disc about an axis through A and ⊥ar to the plane of disc = $I = \frac{1}{2} m a^2$

Angular momentum before addition of particle
$$= I\omega = \frac{1}{2} m\omega a^2$$

At the instant when the particle is placed on the disc a pair of equal and opposite frictional impulses act, one on the disc and one on the particle causing equal and opposite changes in momentum about the axis through A. Hence there is no net impulse and no net change in the angular momentum about A.

The particle stays in contact with a fixed point on the disc so it has the same angular velocity as the disc.

M.I of the particle and disc about the same axis $= \frac{ma^2}{2} + 2m\left(\frac{a}{2}\right)^2$

$$= \frac{ma^2}{2} + \frac{ma^2}{2} = ma^2$$

Let ω_1 be the velocity of (particle + disc). Then angular momentum of the system $= ma^2\omega_1$

By Law of Conservation of momentum

$$\frac{ma^2}{2}\omega = ma^2\omega_1$$
$$\omega_1 = \frac{\omega}{2}$$

Thus the system rotates with an angular velocity $\frac{\omega}{2}$

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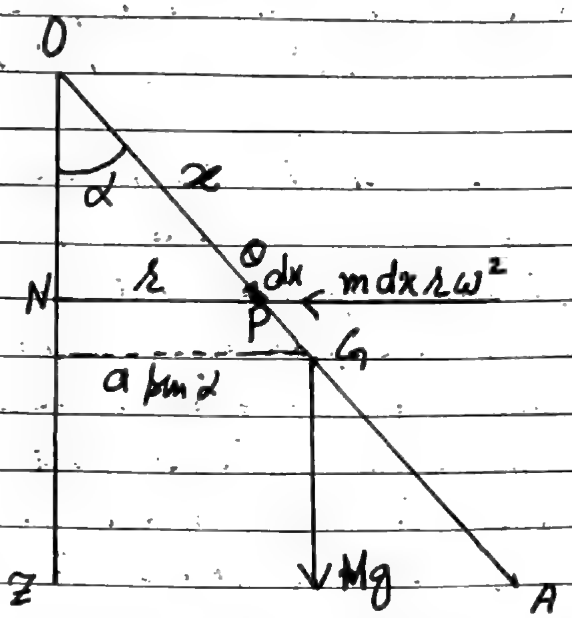
Problem # A uniform rod OA, of mass M and length 2a, is free to turn about a fixed hinge at one end O and revolves about a vertical line OZ so as to describe a cone of semi-vertical angle α, to find the angular velocity.

Sol*



Let m be the mass per unit length of the rod so that

$$m = \frac{M}{2a} \Rightarrow M = 2am$$

and consider an element $PQ = dx$ at a distance x from the fixed point O

Draw perpendicular PN to OZ. Since the element PQ describes a circle of radius r about N, its acc towards N is $r\omega^2$

where $r = PN = x\sin\alpha$

Mass of the element $dx = m\,dx$

The force towards N on $dx = m\,dx \cdot r\omega^2$

The moment of this force about O

$= $ force $\times$ moment arm

$= m\,dx \cdot r\omega^2 \times ON$ $\qquad ON = x\cos\alpha$

$= m\,dx \cdot x\sin\alpha \cdot \omega^2 \cdot x\cos\alpha$

$= m\omega^2 \sin\alpha\cos\alpha \cdot x^2\,dx$

Moment of the whole rod about O

$= \int_0^{2a} m\omega^2 \sin\alpha\cos\alpha\, x^2\,dx = $ moment of weight

$$\int_0^{2a} m\omega^2 \sin\alpha \cos\alpha \, x^2 dx = Mg \cdot a \sin\alpha$$

$$m\omega^2 \sin\alpha \cos\alpha \left| \frac{x^3}{3} \right|_0^{2a} = Mg \cdot a \sin\alpha$$

$$m\omega^2 \sin\alpha \cos\alpha \cdot \frac{8a^3}{3} = Mg \cdot a \sin\alpha$$

Putting $M = 2am$.

$$m\omega^2 \sin\alpha \cos\alpha \cdot \frac{8a^3}{3} = 2amg \cdot a \sin\alpha$$

$$4m\omega^2 a \sin\alpha \cos\alpha = 3g \sin\alpha$$

$$4m\omega^2 a \sin\alpha \cos\alpha - 3g \sin\alpha = 0$$

$$\sin\alpha(4m\omega^2 \cos\alpha - 3g) \sin\alpha = 0$$

$\Rightarrow \sin\alpha = 0$ or $4m\omega^2 \cos\alpha - 3g = 0$

$\Rightarrow \alpha = 0$ or $4m\omega^2 \cos\alpha = 3g$

i.e rods hangs vertically

$$\cos\alpha = \frac{3g}{4a\omega^2}$$

or $\cos\alpha = \dfrac{3g}{4a\omega^2}$

when $\alpha \neq 0$, then $\cos\alpha < 1$

$$\Rightarrow 4a\omega^2 > 3g$$

i.e $\omega^2 > \dfrac{3g}{4a}$

$$\omega > \sqrt{\frac{3g}{4a}}$$

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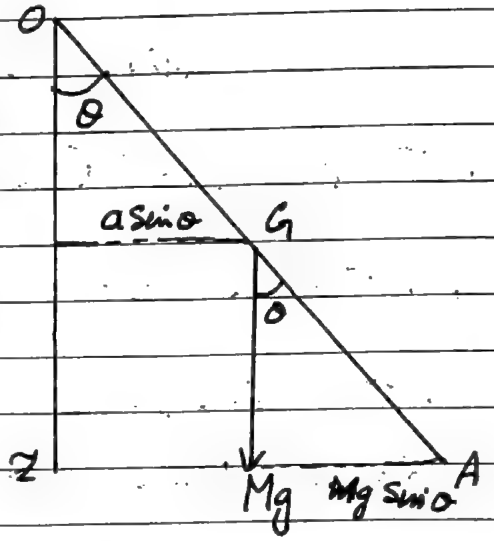
Problem# A uniform rod OA, of mass M and length 2a can turn freely about one end O which is fixed. It started with an angular velocity ω from the position in which it hangs vertically. To find the least value of ω in order that the rod may make complete revolutions.



Sol# M.I of rod about an axis through end O and ⊥ to rod

$$= \frac{4}{3} M a^2 = I$$

Because M.I about centroidal axis ⊥ar to rod (here normal to rod) $= \frac{1}{3} Ma^2$ and

By parallel axis Theorem M.I about an axis through O and normal to rod

$$= \frac{1}{3} Ma^2 + Ma^2 = \frac{4}{3} Ma^2$$

Equation of motion is

$$I\alpha = \text{Moment of the force}$$

$$I \frac{d^2\theta}{dt^2} = - Mg \sin\theta . a$$

-ve sign because force is opposite to motion of rod

$$\frac{4}{3} Ma^2 \ddot{\theta} = - Mg a \sin\theta$$

$$\ddot{\theta} = - \frac{3g}{4a} \sin\theta$$

Multiplying by $2\frac{d\theta}{dt} = 2\dot{\theta}$ and integrating.

$$\int \frac{d}{dt}(\dot{\theta})^2 = -\frac{3g}{4a}\int \sin\theta \cdot 2\frac{d\theta}{dt}\, dt$$

$$\dot{\theta}^2 = \frac{3g}{2a}\cos\theta + C \rightarrow (1)$$

when $\dot{\theta} = \omega$ $\quad \theta = 0$

$$\omega^2 = \frac{3g}{2a}\cos 0^\circ + C$$

$$\omega^2 = \frac{3g}{2a} + C$$

$$C = \omega^2 - \frac{3g}{2a}$$

using in (1)

$$\dot{\theta}^2 = \frac{3g}{2a}\cos\theta + \omega^2 - \frac{3g}{2a}$$

$$= \omega^2 - \frac{3g}{2a}(1-\cos\theta)$$

Now for Complete revolution of rod $\theta = \pi$

Now when $\theta = \pi$

$$\dot{\theta}^2 = \omega^2 - \frac{3g}{2a}(2) = \omega^2 - \frac{3g}{a}$$

In this position $\dot{\theta}$ is +ve if $\omega^2 - \frac{3g}{a} > 0$

if $\omega^2 > \frac{3g}{a}$

Hence least value of ω for the rod to make Complete revolutions is $\sqrt{3g/a}$

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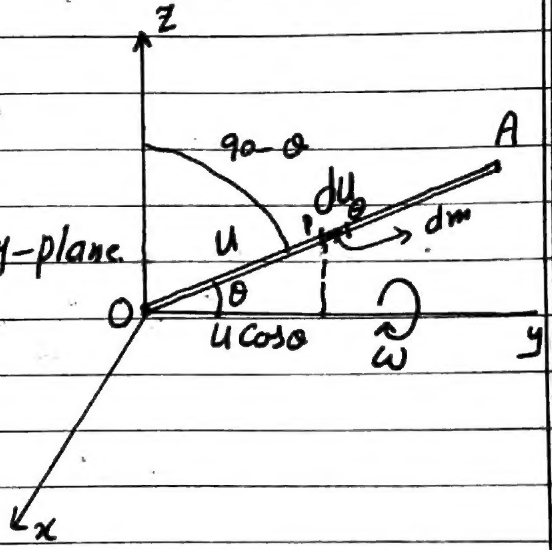
Problem A uniform rod of length l, mass m rotates about the y-axis as an element of a right circular cone. If the angular velocity about the y-axis is ω, determine the expression for for the angular momentum of the rod with respect to x-y-z axes for the position position of rod directly above y-axis with angle θ with y-axis. Also write expression of K.E in this position.

Sol# $\because$ The rod makes angle θ directly above y-axis

$\therefore$ rod will be in zy-plane.

and its angle with z-axis is $90^\circ - \theta$.



Let ρ be mass density of rod. Then.

$$\rho = \frac{m}{l}$$

Consider mass element dm at PQ = du at distance u from O.

P.V of dm relative O is

$$\underline{r} = u\cos\theta\,\hat{j} + u\cos(90-\theta)\hat{k}$$

$$\underline{r} = u\cos\theta\,\hat{j} + u\sin\theta\,\hat{k}$$

Also $\underline{\omega} = 0\hat{i} + \omega\hat{j} + 0\hat{k}$

$\therefore$ Velocity $\underline{v}$ of dm is

$$\underline{v} = \underline{\omega} \times \underline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega & 0 \\ u\cos\theta & u\cos\theta & u\sin\theta \end{vmatrix}$$

$$= u\omega\sin\theta\,\hat{i}$$

$$= u\omega\sin\theta\,\hat{i} + 0\hat{j} + 0\hat{k}$$

Angular momentum of mass element dm is

$$d\underline{L}_o = \underline{r} \times dm\,\underline{v}$$

$$= dm\; \underline{r} \times \underline{v}$$

$$= dm \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & u\cos\theta & u\sin\theta \\ u\omega\sin\theta & 0 & 0 \end{vmatrix}$$

$$= dm\left[0\hat{i} + \hat{j}\,u^2\omega\sin^2\theta - u^2\omega\sin\theta\cos\theta\,\hat{k}\right]$$

$$= \left(u^2\omega\sin^2\theta\,\hat{j} - u^2\omega\sin\theta\cos\theta\,\hat{k}\right)dm$$

$$= \left(\sin\theta\,\hat{j} - \cos\theta\,\hat{k}\right)u^2\omega\sin\theta\,dm$$

$$dm = \rho\,du$$

$$d\underline{L}_o = \left(\sin\theta\,\hat{j} - \cos\theta\,\hat{k}\right)u^2\omega\sin\theta\,\rho\,du$$

Angular momentum of the whole rod is given by

$$\underline{L}_o = \int_0^l \left(\sin\theta\,\hat{j} - \cos\theta\,\hat{k}\right)\omega\sin\theta\,\rho\cdot u^2\,du$$

$$= \left(\sin\theta\,\hat{j} - \cos\theta\,\hat{k}\right)\omega\sin\theta\,\rho\cdot\left|\frac{u^3}{3}\right|_0^l$$

$$= \omega\sin\theta\left(\sin\theta\,\hat{j} - \cos\theta\,\hat{k}\right)\cdot\frac{m}{l}\,\frac{l^3}{3}$$

$$= \frac{1}{3}ml^2\omega\sin\theta\left(\sin\theta\,\hat{i} - \cos\theta\,\hat{k}\right)$$

Expression for K.E

$$K.E = \frac{1}{2}I_y\omega^2$$

Now we find M.I, I_y of rod about y-axis

M.I of mass element dm about y-axis

$$= dI_y = (u\sin\theta)^2 dm$$

$$= u^2\sin^2\theta \cdot \rho\, du$$

$$I_y = \int_0^l \sin^2\theta \cdot \rho\, u^2 du$$

$$= \rho \sin^2\theta \left[\frac{u^3}{3}\right]_0^l$$

$$= \frac{m}{l}\sin^2\theta \cdot \frac{l^3}{3}$$

$$= \frac{ml^2}{3}\sin^2\theta$$

$$K.E = \frac{1}{2} I_y \omega^2 = \frac{1}{6} ml^2\omega^2\sin^2\theta$$

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